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Functions of classes L_p and Lip (α, p) and their Riesz means (**)

1 - Definitions and notations

Let $f \in L(-\pi, \pi)$ and be periodic with period 2π. Let the Fourier series of f at x be given by

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.1}$$

Let (p_n) be a sequence of non-negative constants such that

$$P_n = p_0 + p_1 + \ldots + p_n \neq 0 \qquad (n \geqslant 0).$$

Then the transformation

$$t_n = (P_n)^{-1} \sum_{m=0}^{n} p_m s_m \tag{1.2}$$

where (s_n) is the sequence of partial sums of the series $\sum_{n=0}^{\infty} c_n$ of real numbers, are called the *Riesz means* (R, p_n) or simply (R, p_n) *means* of (s_n).

A function $f \in \text{Lip}\,\alpha$ $(\alpha > 0)$ if

$$f(x+h) - f(x) = O(|h|^\alpha) \qquad (h \to 0) \tag{1.3}$$

and $f \in \text{Lip}(\alpha, p)$ $(\alpha > 0,\ p \geqslant 1)$ if (see [2], p. 612)

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p \, dx\right)^{1/p} = O(|h|^\alpha). \tag{1.4}$$

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Further, if $f \in L_p$ $(p \geq 1)$, the expression

$$\omega_p(\delta; f) = \sup_{0 \leq h \leq \delta} \{\frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p \, dx\}^{1/p}$$

is called *integral modulus of continuity* of f.

Throughout the paper, $R_n(f; x)$ will denote (R, p_n) means of $(s_n(x))$, the partial sum of (1.1). All norms, to be considered in this paper, will be L_p $(p \geq 1)$ norms with respect to the variable x. We also use $\uparrow$ (or $\downarrow$) for non-decreasing (or non-increasing). Some time, for the convenience, we write $P(k)$ for P_k. We also write

(1.5) $$\varphi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

(1.6) $\xi = [\pi/t]$ the integral part of (π/t) in $0 < t \leq \pi$.

2 - Introduction

Hardy and Littlewood [2] have stated without proof that the class of functions $\mathrm{Lip}(\alpha, p)$ is identical with the class of functions approximable in the L_p-norm with an error $O(n^{-\alpha})$ by trigonometrical polynomials of degree n. With a view to examining the range of values of α and p for which the statement of Hardy and Littlewood holds. Quade [5] has obtained the following amongst other results:

Theorem A. In the function $f(x)$ can be approximated for each $n \geq 1$, by a trigonometrical polynomial, $t_n(x)$, of degree n at most, such that $\|f - t_n\|_p = O(n^{-\alpha})$ $p \geq 1$, then

(i) if $0 < \alpha < 1$ $f(x) \in \mathrm{Lip}(\alpha, p)$; (ii) if $\alpha = 1$ $\omega_p(\delta; f) = O\{\delta \log \delta^{-1}\}$.

Moreover there exist functions for which $\|f - t_n\|_p = O(n^{-1})$ which do not belong to $\mathrm{Lip}(1, p)$.

Theorem B. If $f(x) \in \mathrm{Lip}(\alpha, p)$, $p \geq 1$, $0 < \alpha \leq 1$, then, for any integer n, $f(x)$ may be approximated in L_p by a trigonometric polynomial $t_n(x)$ of order n such that

$$\|f - t_n\|_p = O(n^{-\alpha}).$$

With a view to obtaining the degree of approximation of the Riesz-means (R, p_n) to $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leq 1)$, we [1] proved the following

Theorem C. If $f \in \mathrm{Lip}\,\alpha$ $0<\alpha\leq 1$, then

$$\max_{0\leq x\leq 2\pi} |R_n(f;x)-f(x)| = \begin{cases} O\{(p_n/p_n)^\alpha\} & 0<\alpha<1 \\ O\{(p_n/P_n)\log(P_n/p_n)\} & \alpha=1 \end{cases}$$

where $0\leq (p_n) \downarrow$.

The object of this paper is to obtain the degree of convergence of $R_n(f;x)$ to $f(x)$ in the L_p-norm whenever either $f\in L_p (p>1)$ or $f \in \mathrm{Lip}(\alpha, p)$. Precisely, we prove the following

Theorem 1. *Let* $f\in L_p$ $(p>1)$ *and let* $0\leq (p_n) \uparrow$. *Then*

$$\|R_n(f)-f\|_p = O\{(P_n)^{-1}\sum_{k=1}^{n} k^{-1}P_k\,\omega_p(\pi/k;f)\}. \tag{2.1}$$

Theorem 2. *Let* $f\in L_p (p>1)$ *and let* $0\leq (p_n) \downarrow$. *Suppose* $\omega_p(t;f)$ *satisfies the following as* $t\to 0+$

$$\int_t^\pi u^{-2}\omega_p(u;f)\,du = O\{H(t)\} \tag{2.2}$$

where $H\geq 0$ *and that*

$$\text{(i)}\quad tH(t)=o(1) \qquad \text{(ii)}\quad \int_0^t H(u)\,du = O\{tH(t)\}. \tag{2.3}$$

Then

$$\|R_n(f)-f\|_p = O\{(p_n/P_n)H(p_n/P_n)\}. \tag{2.4}$$

Theorem 3. *Let* f *and* (p_n) *be as defined in Theorem* 2 *and let* $\omega_p(t;f)$ *satisfy* (2.2) *and* (2.3) (ii). *Then*

$$\|R_n(f)-f\|_p = O\{(p_n/P_n)H(\pi/n)\}. \tag{2.5}$$

Theorem 4. *If* $f\in \mathrm{Lip}(\alpha,p)$ $0<\alpha\leq 1$, $p>1$ *and if* $(p_n)\geq 0$, *then*

$$\|R_n(f)-f\|_p = O\{(P_n)^{-1}\sum_{k=1}^{n} k^{-1-\alpha}P_k\} \qquad ((p_n)\uparrow) \tag{2.6}$$

and, whenever $(p_n)\downarrow$,

$$(2.7)\qquad \|R_n(f)-f\|_p=\begin{cases}O\{(p_n/P_n)^\alpha\} & 0<\alpha<1\\ O\{(p_n/P_n)\log(P_n/p_n)\} & \alpha=1.\end{cases}$$

Theorem 5. *Let* $f\in \mathrm{Lip}(\alpha,p)$ $0<\alpha\leqslant 1$, $p>1$, $\alpha p>1$ *and let* (p_n) *be non-negative. Then, uniformly in* x *almost everywhere, the following hold*

$$(2.8)\qquad R_n(f;x)-f(x)=\begin{cases}O\{(P_n)^{-1}\sum_{k=1}^{n}k^{1/p-1-\alpha}P_k\} & (p_n)\uparrow\\ O\{(p_n/P_n)^{\alpha-1/p}\} & (p_n)\downarrow.\end{cases}$$

3 - Lemmas

We require the following lemmas in the proof of the theorems.

Lemma 1. *If* $h(x,t)$ *is a function of two variables defined for* $0\leqslant t\leqslant\pi$, $0\leqslant x\leqslant 2\pi$, *then*

$$\|\int h(x,t)\,\mathrm{d}t\|_p\leqslant\int\|h(x,t)\|_p\,\mathrm{d}t\,(p>1).$$

For its proof, see [3] (p. 148, 6.13.9).

Lemma 2. *Suppose that* $f\in\mathrm{Lip}(\alpha,p)$ *where* $p\geqslant 1$, $0<\alpha\leqslant 1$, $\alpha p>1$. *Then* f *is equal to a function* $g\in\mathrm{Lip}(\alpha-1/p)$ *almost everywhere.*

For its proof, see [2] (Theorem 5(ii), p. 627).

Lemma 3. *Let* $0\leqslant(p_n)\uparrow$. *Then uniformly in* $0<t\leqslant\pi$

$$\sum_{k=0}^{n}p_k\sin(k+\frac{1}{2})t=O(P_\tau).$$

This may be obtained by using arguments similar to that of [4] (p. 182).

4 - Proof of the theorems

Proof of Theorem 1. We have

$$(4.1)\qquad R_n(f;x)-f(x)=(\pi P_n)^{-1}\int_0^\pi\{\varphi_x(t)/\sin\frac{1}{2}t\}\{\sum_{k=0}^{n}p_k\sin(k+\frac{1}{2})t\}\,\mathrm{d}t=I_1+I_2.$$

Then, by Minkowski's inequality,

$$\|R_n(f)-f\|_p \leqslant \|I_1\|_p + \|I_2\|_p . \tag{4.2}$$

However, $\sin\frac{1}{2}t \geqslant t/\pi$ $(0 \leqslant t \leqslant \pi)$ and since $\sin(k+\frac{1}{2})t \leqslant (k+1)t$ we have by Lemma 1

$$\|I_1\|_p \leqslant (P_n)^{-1} \int_0^{\pi/n} t^{-1} \|\varphi(t)\|_p \,|\sum_{k=0}^{n} p_k \sin(k+\frac{1}{2})t|\,dt \leqslant (n+1) \int_0^{\pi/n} \|\varphi(t)\|_p\,dt$$

where, by Minkowski inequality and the periodicity of f,

$$\|\varphi(t)\|_p = \{\frac{1}{2\pi}\int_0^{2\pi} |\varphi_x(t)|^p\,dx\}^{1/p}$$

$$\leqslant \frac{1}{2}\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+t)-f(x)|^p\,dx\}^{1/p} + \frac{1}{2}\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)-f(x-t)|^p\,dx\}^{1/p}$$

$$\leqslant \omega_p(t;f) .$$

Hence, since $\omega_p(t;f)$ is non-decreasing with t, we get

$$\|I_1\|_p \leqslant 2\pi\omega_p(\pi/n;f) \leqslant (2\pi/P_n) \sum_{k=1}^{n} k^{-1} P_k\,\omega_p(\pi/k;f)$$

since $(p_n)\uparrow$. Also, by Lemma 3, we have

$$\|I_2\|_p = O(1/P_n) \int_{\pi/n}^{\pi} t^{-1}\omega_p(t;f)\,P([\pi/t])\,dt = O(1/P_n)\sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} t^{-1}\omega_p(t;f)\,P([\pi/t])\,dt$$

$$= O(1/P_n)\sum_{k=1}^{n} k^{-1}P_k\,\omega_p(\pi/k;f) .$$

Using the estimates of $\|I_1\|_p$ and $\|I_2\|_p$ in (4.2), we get (2.1).

Proof of Theorem 2. We have

$$R_n(f;x)-f(x) = (1/\pi P_n)\,(\int_0^{p_n/P_n} + \int_{p_n/P_n}^{\pi})\,\frac{\varphi_x(t)}{\sin\frac{1}{2}t}\{\sum_{k=0}^{n} p_k \sin(k+\frac{1}{2})t)\,dt = J_1 + J_2 . \tag{4.3}$$

Then, by Minkowski's inequality,

$$\|R_n(f)-f\|_p \leqslant \|J_1\|_p + \|J_2\|_p$$

where

$$\|J_1\|_p \leqslant \int_0^{p_n/P_n} t^{-1}\omega_p(t;f)\,\mathrm{d}t = [-t\int_t^{\pi} u^{-2}\omega_p(u;f)\,\mathrm{d}u]_0^{p_n/P_n} + \int_0^{p_n/P_n} \mathrm{d}t \int_t^{\pi} u^{-2}\omega_p(u;f)\,\mathrm{d}u$$

$$= O(p_n P_n)\,H(p_n/P_n) + O(1)\int_0^{p_n/P_n} H(t)\,\mathrm{d}t = O\{(p_n/P_n)\,H(p_n/P_n)\}$$

by (2.2) and (2.3). And since $0 \leqslant (p_n) \downarrow$, we obtain, by Lemma 1 and Abel's Lemma and by (2.2)

$$\|J_2\|_p = O\{(p_n/P_n)\int_{p_n/P_n}^{\pi} t^{-2}\omega_p(t;f)\,\mathrm{d}t\} = O\{(p_n/P_n)\,H(p_n/P_n)\}\,.$$

Using the estimates $\|J_1\|_p$ and $\|J_2\|_p$, we may get (2.4).

Proof of Theorem 3. Proceeding as in Theorem 1, we have

$$\|I_1\|_p \leqslant (n+1)\int_0^{\pi/n} \omega_p(t;f)\,\mathrm{d}t$$

where integration by parts yields that

$$\int_0^{\pi/n} \omega_p(t;f)\,\mathrm{d}t = [-t^2\int_t^{\pi} u^{-2}\omega_p(u;f)\,\mathrm{d}u]_0^{\pi/n} + 2\int_0^{\pi/n} t\int_t^{\pi} u^{-2}\omega_p(u;f)\,\mathrm{d}u$$

$$= O\{n^{-2}H(\pi/n)\}$$

by (2.2) and (2.3) (ii), and hence

$$\|I_1\|_p = O\{(p_n/P_n)\,H(\pi/n)\}$$

since $(n+1)p_n \geqslant P_n$. Also by Abel's lemma

$$\|I_2\|_p = O\{(p_n/P_n)\int_{\pi/n}^{\pi} t^{-2}\omega_p(t;f)\,\mathrm{d}t\} = O\{(p_n/P_n)\,H(\pi/n)\}$$

by (2.2). Hence the proof of (2.5) may be completed.

Proof of Theorem 4. Since $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leqslant 1$, $p > 1$, implies that

$$\omega_p(t;f) = O(t^{\alpha})\,, \tag{4.4}$$

we get (2.6) by using (4.4) in (2.1). Further, by using (4.4) in (2.2), we get

$$H(t) = \begin{cases} O(t^{\alpha-1}) & 0<\alpha<1 \\ O(\log \frac{\pi}{t}) & \alpha=1. \end{cases} \tag{4.5}$$

Finally, by using (4.5) in (2.4) with $t=p_n/P_n$, we get (2.7).

Proof of Theorem 5. By Lemma 2 the hypothesis $f \in \mathrm{Lip}(\alpha, p)$, where $p>1$, $0<\alpha\leq 1$, $\alpha p>1$, implies that there exists a function $g \in \mathrm{Lip}(\alpha - 1/p)$ such that

$$f=g \qquad \text{almost every where.}$$

Hence we can conclude that

$$\varphi_x(t) = O(t^{\alpha-1/p}) \qquad \text{almost every where.} \tag{4.6}$$

We first consider the case when $(p_n) \uparrow$. Using the notation of (4.1), we have

$$I_1 \leq (n+1) \int_0^{\pi/n} |\varphi_x(t)|\, dt = O(n^{1/p-\alpha}) = O(1/P_n) \sum_{k=1}^{n} k^{1/p-1-\alpha} P_k$$

by (4.6) and the fact that $(P_n/n) \uparrow$. By Lemma 3 and (4.6)

$$I_2 = O(1/P_n) \int_{\pi/n}^{\pi} t^{-1/p-1+\alpha} P([\pi/t])\, dt = O(1/P_n) \sum_{k=1}^{n} h^{1/p-1-\alpha} P_k.$$

Combining I_1 and I_2, we get (2.8) in the case when $(p_n) \uparrow$.
Now, we consider the case $(p_n) \downarrow$. Using the notation of (4.3), we get by (4.6)

$$J_1 = O(1) \int_0^{p_n/P_n} t^{-1} |\varphi_x(t)|\, dt = O\{(p_n/P_n)^{\alpha-1/p}\}$$

and by Abel's Lemma and (4.6)

$$J_2 = O(p_n/P_n) \int_{p_n/P_n}^{\pi} t^{-2} |\varphi_x(t)|\, dt = O\{(p_n/P_n)^{\alpha-1/p}\}.$$

Combining J_1 and J_2, we get (2.8).
This completes the proof of Theorem 5.

References

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Summary

The object of this paper is to obtain the degree of convergence of $R_n(f;x)$ *to* $f(x)$ *in the* L_p*-norm whenever either* $f \in L_p$ $(p > 1)$ *or* $f \in \mathrm{Lip}(\alpha, p)$.

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