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Change of phase with variable melting temperature (**)

1 - Introduction

Mathematical problem describing change of phase in materials in which the temperature $\bar{\theta}$ at which the phase transition occurs is variable have been considered e.g.[1], [2], [3]. Here, we assume that $\bar{\theta}$ depends on space and on time. Referring to a model problem in one space dimension, we will discriminate between cases in which a *mushy region* (i.e. a region where the temperature θ is exactly equal to $\bar{\theta}$) appears and cases in which the domain under consideration is divided in two regions S and M, where $\theta < \bar{\theta}$ and $\theta = \bar{\theta}$ respectively.

We assume that thermal capacity is constant and that the conductivity is k_L for $\theta > \bar{\theta}$ and k_S for $\theta < \bar{\theta}$.

Defining

$$E = \int_{\bar{\theta}(x,t)}^{\theta(x,t)} c\,dz + \lambda sgn^{+}(\theta - \bar{\theta}), \tag{1.1}$$

the thermal balance equation can be written formally as

$$\frac{\partial E}{\partial t} + c\frac{\partial \bar{\theta}}{\partial t} - \frac{\partial}{\partial x}(k\frac{\partial \theta}{\partial x}) = 0 \tag{1.2}$$

and the definition of a weak solution can be given in standard way. It has to be noted, as in [2], that the values of the conductivity are to be assigned as function

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of E; otherwise uniqueness can not be guaranteed (if there exists a mushy region). Here, we assume $k = k_L$ for $E > \lambda$, $k = k_S$ for $E < 0$ (as already stated) and in addition $k = k_S + \alpha E$ $(\alpha = (k_L - k_S)/\lambda,\ E \in [0, \lambda])$.

We will consider a problem in the region $(0, 1) \times \boldsymbol{R}^+$ with the following initial and boundary conditions

(1.3) $\theta(x, 0) = h(x)$ (1.4) $\theta_x(0, t) = 0$ (1.5) $\theta(1, t) = g(t)$

and assume that it admits a classical solution in the sense of [2]. Namely we assume that three smooth regions can be defined (which will be called liquid, solid and mushy region, respectively)

$$L = \{(x, t) : \theta(x, t) > \bar{\theta}(x, t)\} \qquad S = \{(x, t) : \theta(x, t) < \bar{\theta}(x, t)\}$$

$$M = \{(x, t) : \theta(x, t) = \bar{\theta}(x, t)\}$$

such that

$$\theta_t(x, t) - k_S \theta_{xx}(x, t) = 0 \qquad (x, t) \in S \tag{1.6}$$

$$\theta_t(x, t) - k_L \theta_{xx}(x, t) = 0 \qquad (x, t) \in L \tag{1.7}$$

$$E_t(x, t) + c\bar{\theta}_t(x, t) - [k\bar{\theta}_x(x, t)]_x = 0 \qquad (x, t) \in M\ . \tag{1.8}$$

Furthermore, the interphase conditions are:

$$\lambda \dot{s}(t) = -k_L \theta_x^L + k_S \theta_x^S \tag{1.9}$$

if $x = s(t)$ is the interpahse between S and L;

$$E(s(t)-, t)\,[\dot{s}(t) + \alpha\bar{\theta}_x(s(t), t)] + k_S[\bar{\theta}_x(s(t), t) - \theta_x(s(t)+, t)] = 0 \tag{1.10}$$

if $x = s(t)$ is the interphase between S and M (say, S lies on the right);

$$[E(s(t)+, t) - \lambda]\,[\dot{s}(t) + \alpha\bar{\theta}_x(s(t), t)] + k_L[\bar{\theta}_x(s(t), t) - \theta_x(s(t)-, t)] = 0 \tag{1.11}$$

if $x = s(t)$ is the interphase between M and L (say, L lies on the left).

We note that if meas $M = 0$, then the problem is of Stefan type. In the next section we will investigate whether this situation appears or not depending on the data.

2 - Appearance of a mushy region

We will assume

$$h(x) < \bar{\theta}(x, 0) \qquad 0 < x \leqslant 1, \tag{2.1}$$

$$g(t) < \bar{\theta}(1, t) \qquad 0 \leqslant t, \tag{2.2}$$

$$\bar{\theta}_x(0, t) = 0 \qquad 0 \leqslant t. \tag{2.3}$$

We have

Proposition 2.1. *Assume either* $h(0) < \bar{\theta}(0, 0)$ *or*

$$k_S \bar{\theta}_{xx}(x, t) - \bar{\theta}_t(x, t) \leqslant 0 \qquad x \in (0, \delta_1) \quad t \in (0, t_1)\ (^1).$$

Then there exists $t_2 > 0$ *such that* $(0, 1) \times (0, t_2) \subset S$.

Proof. The proposition follows at once by the continuity of the solution in the problem corresponding to data (1.3)-(1.5) and by straightforward application of the maximum principle.

Since, in cases above, the problem is trivial (up to t_2), we will assume henceforth

$$h(0) = \bar{\theta}(0, 0) \tag{2.4}$$

$$k_S \bar{\theta}_{xx} - \bar{\theta}_t \geqslant 0 \qquad 0 \leqslant x \leqslant 1 \quad t \geqslant 0. \tag{2.5}$$

Inequality (2.5) was assumed to hold in the whole strip $(0, 1) \times \boldsymbol{R}^+$ for sake of simplicity. Actually the assumption we do really need is that $k_S \bar{\theta}_{xx} - \bar{\theta}_t$ (and $k_L \bar{\theta}_{xx} - \bar{\theta}_t$) has a definite sign in a neighborhood of the origin (0, 0). Indeed our analysis is only local.

Without loss of generality, we will set $h(0) = \bar{\theta}(0, 0) = 0$.

We have the following

Theorem 2.2. *Assume* (2.1)-(2.5) *and either*

(1) Henceforth, we denote by $t_1, t_2, \ldots, \delta_1, \delta_2, \ldots$ as appropriate positive constants and θ, h, g will be assumed to be as smooth as we will need.

$$(\mathrm{C})_1 \qquad h'(0)<0\,, \qquad or$$

$$(\mathrm{C})_2 \qquad h'(0)=0 \qquad k_S h''(0)<\bar{\theta}_t(0,0)\,, \qquad or$$

$$(\mathrm{C})_3 \qquad h'(0)=0 \qquad k_S h''(0)=\bar{\theta}_t(0,0) \qquad k_S h'''(0)<0\,.$$

Then, there exists a function $\theta(x,t)$ *and time* $t_3>0$ *such that* $(0,1)\times(0,t_3)\subset S$, *i.e. the solution of*

$$(\mathrm{P})' \qquad \begin{aligned} &k_S\theta_{xx}-\theta_t=0 \qquad 0<x<1 \qquad 0<t \\ &\theta(x,0)=h(x) \quad 0\leqslant x\leqslant 1\,, \quad \theta_x(0,t)=0 \quad 0<t\,, \quad \theta(1,t)=g(t) \quad 0<t \end{aligned}$$

satisfies the inequality

$$(2.6) \qquad \theta(x,t)<\bar{\theta}(x,t) \qquad 0\leqslant x\leqslant 1 \qquad 0<t<t_3\,.$$

Proof. *Case* $(\mathrm{C})_1$. There exists a smooth function $\tilde{h}(x)$ such that

$$h(x)\leqslant\tilde{h}(x)\leqslant\bar{\theta}(x,0) \qquad 0\leqslant x\leqslant 1$$
$$\tilde{h}'(0)=\bar{\theta}_x(0,0)=0 \qquad \tilde{h}''(0)<\theta_t(0,0)/k_S\,.$$

Then we consider the problem

$$(\tilde{\mathrm{P}})' \qquad \begin{aligned} &k_S\tilde{\theta}_{xx}-\tilde{\theta}_t=0 \qquad 0<x<1 \qquad 0<t \\ &\tilde{\theta}(x,0)=\tilde{h}(x) \quad 0\leqslant x\leqslant 1\,, \quad \tilde{\theta}_x(0,t)=0 \quad 0<t\,, \quad \tilde{\theta}(1,t)=g(t) \quad 0<t\,. \end{aligned}$$

Obviously, we get

$$\theta(x,t)\leqslant\tilde{\theta}(x,t) \qquad 0\leqslant x\leqslant 1 \qquad 0\leqslant t\,.$$

Since $\tilde{\theta}$ satisfies assumptions $(\mathrm{C})_2$. We reduce to the case below.

Case $(\mathrm{C})_2$. We set

$$(2.7) \qquad U(x,t)=\theta(x,t)-\bar{\theta}_t(0,0)\,t+\varepsilon t-h(x)\,,$$

and have

$$k_S U_{xx} - U_t = \bar{\theta}_t(0,0) - k_S h''(0) - \varepsilon + O(x) > 0 \qquad 0 < x < \delta_1 \qquad 0 < t < t_1$$
$$U(x,0) = 0 \qquad U_x(0,t) = 0 \qquad U_t(\delta, t) = k_S \theta_{xx}(\delta, t) - \bar{\theta}_t(0,0) + \varepsilon.$$

By means of the continuity of $h(x)$ and $\theta(x,t)$ there exists $\delta_4, t_4 > 0$ such that $U(\delta_4, t) < 0 \quad 0 \leqslant t \leqslant t_4$, and hence

$$U(x,t) < 0 \qquad 0 \leqslant x \leqslant \delta_4, \quad 0 < t < t_4,$$

i.e. $$\theta(x,t) < \bar{\theta}_t(0,0)t - \varepsilon t + h(x) = \bar{\theta}(x,t) - \bar{\theta}(x,0) + h(x) - \varepsilon t - o(t^2 + tx) < \bar{\theta}(x,t)$$
$$0 \leqslant x \leqslant \delta_4 \qquad 0 < t < t_4.$$

Case $(\mathrm{C})_3$. We set

$$U(x,t) = \theta(x,t) - \bar{\theta}_t(0,0)t + \varepsilon xt - h(x) \tag{2.8}$$

and can prove $\theta(x,t) < \bar{\theta}(x,t) \qquad 0 \leqslant x \leqslant \delta_5 \qquad 0 < t < t_5$ in the same way.

Remark 2.3. According to our proof, we can discuss the case in which $k_S h'''(0) = \bar{\theta}_{xt}(0,0) = 0$ in the same way.

Remark 2.4. If the assumption (2.3) does not hold and $\bar{\theta}_x(0,t) \leqslant 0$, Theorem 2.2 can be still proved as $-\bar{\theta}_{xt}(0,0)xt$ is added to the right hand of (2.8). But if $\bar{\theta}_x(0,t) < 0$, we need the condition $h'(0) \leqslant 0$ in case $(\mathrm{C})_2$ or $h'(0) < 0$ or $h'(0) = 0$ and $\bar{\theta}_{xt}(0,0) \leqslant 0$ in case $(\mathrm{C})_3$.

Essentially, Theorem 2.2 states that under conditions $(\mathrm{C})_1$, $(\mathrm{C})_2$ or $(\mathrm{C})_3$, the phase-change does not begin at $t = 0$. Thus the heat conduction problem is trivially solvable until these conditions are violated. Thus we assume now, besides (2.1)-(2.5), that either

$(\mathrm{d})_1$ $$h'(0) = 0 \qquad k_S h''(0) > \bar{\theta}_t(0,0), \qquad \text{or}$$

$(\mathrm{d})_2$ $$h'(0) = 0 \qquad k_S h''(0) = \bar{\theta}_t(0,0) \qquad k_S h'''(0) > 0,$$

and we have

Theorem 2.5. *Under the assumptions above, for any* $t_1 \in \boldsymbol{R}^+$, *we have* $(0, t_1) \times (0,1) \not\subset S$.

Proof. The proof follows essentially the same lines of the proof of Theorem 2.2.

Thus, in the assumptions $(d)_1$ or $(d)_2$, another phase develops from the very beginning of the process. According to our assumptions on the existence of a classical solution to the phase-change problem, we have that there exists a function $s(t) \in C'(\boldsymbol{R}^+)$, $0 \leqslant s(t) \leqslant 1$, $s(t) \not\equiv 0$ in any neighborood of $t=0$. This is an easy consequence of Theorem 2.5 and of the maximum principle.

Now we prove

Theorem 2.6. *In the assumptions of Theorem* 2.5 *there exist* $t_2 > 0$ *and a function* $\sigma(t)$ *satisfying* $0 \leqslant \sigma(t) \leqslant s(t)$ *and* $\sigma(t) \not\equiv s(t)$ *in any neighborhood of* $t=0$ *such that*

$$\theta(x,t) \equiv \bar{\theta}(x,t) \qquad \sigma(t) \leqslant x \leqslant s(t) \quad 0<t<t_2 .$$

Proof. Suppose that there exist $\rho(t) \geqslant 0$, $\rho(0)=0$ and a time t_3, such that in the region

$$R = \{(x,t): \rho(t) < x < s(t), \quad 0<t<t_3\}$$

it is $\theta(x,t) > \bar{\theta}(x,t)$. This is impossible, because of the maximum principle if

$$k_L \bar{\theta}_{xx} - \bar{\theta}_t \leqslant 0 \tag{2.9}$$

in a neighborhood of origin $(0,0)$. So if (2.9) holds, the theorem is proved. Therefore, we will assume

$$k_L \bar{\theta}_{xx} - \bar{\theta}_t \geqslant 0 \tag{2.10}$$

and we will assume (2.10) holds in the whole strip $(0,1) \times \boldsymbol{R}^+$ as we did for (2.5).

Next, we note that the region

$$R^* = \{(x,t): 0 < x < \rho(t), \quad 0<t<t_2\}$$

should belong to M. In fact, no other components of S can exist because of the maximum principle.

Now, we show that $\rho(t) \equiv 0$. As a matter of fact, from (1.11) and (2.10), using Vyborny-Friedman theorem, we obtain $\dot{\rho}(t) + \alpha \bar{\theta}_x(\sigma(t), t) < 0$, $0 < t < t_3$ and note $\bar{\theta}_x(0,t) = 0$; therefore $\dot{\rho}(t) < k_1 \rho(t)$, $0<t<t_3$ where k_1 is dependent on α and max

$|\bar{\theta}_{xx}(x, t)|$. So $\rho(t) < 0$, $0 < t < t_3$. This contradicts $\rho(t) \geq 0$, $0 < t < t_3$. Thus $R^* = \phi$ and we have

$$R = \{(x, t) : 0 < x < s(t), \quad 0 < t < t_3\}.$$

To complete the proof, we show that assuming $R \subset L$ leads to a contradiction again.

Using Green's identity in the region

$$R_t = \{(x, t) : 0 < x < s(\tau), \quad 0 < \tau < t\} \qquad t < t_3$$

we obtain

$$\begin{aligned} 0 &= \int\int_{R_t} (k_L \theta_{xx} - \theta_t) \mathrm{d}x \mathrm{d}t \\ &= \int_0^t k_L \theta_x(s(\tau) - , \tau) \mathrm{d}\tau + \int_0^{s(t)} (\bar{\theta}(x, s^{-1}(x)) - \theta(x, t)) \mathrm{d}x . \end{aligned}$$

Owing to (2.3) and (1.9), we have

$$\begin{aligned} \lambda s(t) &\leq k_1 \int_0^t s(\tau) \mathrm{d}\tau + \int_0^{s(t)} \bar{\theta}(x, s^{-1}(x)) - \bar{\theta}(x, t) \mathrm{d}x \\ &\leq k_1 \int_0^t s(\tau) \mathrm{d}\tau + k_2 \int_0^{s(t)} (t - s^{-1}(x)) \mathrm{d}x \\ &\leq (k_1 + k_2) \int_0^t s(\tau) \mathrm{d}\tau \end{aligned}$$

where $0 < t < t_3$, $k_2 = \max |\bar{\theta}_t(x, t)|$.

Consequently, $s(t) \equiv 0$ $0 < t < t_3$ which is a contradiction to Theorem 2.5.

Finally, we prove that there exists $t_4 > 0$, such that $L \cap \{t \leq t_4\} = \phi$ i.e. we have

Theorem 2.7. *In the assumptions of Theorem* 2.5 *there exists* $t_4 > 0$ *such that* $\sigma(t) \equiv 0$, $0 < t < t_4$.

Proof. Suppose that this theorem is not true, as the proof of Theorem 2.6, the region

$$\{(x, t) : 0 < x < \sigma(t), \quad 0 < t < t_4\}$$

can not be the other region, except for mushy region.

References

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Sommario

Nel problema unidimensionale di cambiamento di fase con temperatura dipendente dallo spazio e dal tempo, si individua quando la «mushy region» si presenta e quando non si presenta.

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