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Wave propagation in heat conducting dielectric solids with thermal relaxation and temperature dependent electric permittivity (**)

1 - Introduction

It is well-known that a physically unrealistic feature of the parabolic Fourier's law of heat conduction is that thermal pulses propagate at infinite speed. So, in recent years, there has been considerable interest in thermodynamic theories (in the framework of extended linear irreversible thermodynamics, but also from a different point of view) accounting for a finite speed of propagation [**4**], [**6**], [**8**], [**12**], [**15**], [**16**], [**18**]$_3$.

In particular, an alternative and efficient approach to incorporate relaxation phenomena in a thermodynamic theory is based on the notion of internal variables: precisely in [**14**] and in subsequent papers, the usefulness of this tool has been demonstrated in order to allow wave propagation in heat conducting viscous fluids and in [**13**] and references therein, the notion of electromagnetic hidden variables has been introduced to describe both elastic ferroelectrics and ferromagnets.

The purpose of this work is to deliver a theory of wave propagation through heat conducting dielectric solids with thermal relaxation within the context of extended linear irreversible thermodynamics [**9**], [**16**].

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A new fundamental relation between the free enthalpy density ζ and the state variables is proposed: it is assumed that ζ, which in ordinary irreversible thermodynamics depends at most on $\theta, \boldsymbol{E}, \mathbf{H}$ (respectively temperature and electromagnetic fields), may, in non-equilibrium, also depend on the heat flux $\boldsymbol{q}$. The model at hand is then specialized by a particular free enthalpy functional so that the linear phenomenological law relating $\boldsymbol{q}$ to the temperature gradient $\boldsymbol{g}$ is just the constitutive equation proposed by Cattaneo [**2**] and the classical costitutive equations of an electromagnetic theory are derived in the hypothesis (consistently with practical applications, [**10**]) that the dielectric coefficient ε is a positive function of θ.

Throughout the paper, we deal with a system in which no electric relaxation phenomena take place and further we suppose that the current density vector $\mathscr{I}$ may be neglected in Maxwell's equations.

In **2** the thermodynamic theory of the aforesaid model of heat conducting dielectric is elaborated and the basic equations are set up.

In **3** we derive the propagation condition and we elucidate both the effects of the dependence of ε on θ and the influence of thermal relaxation on electromagnetic waves.

Besides the possibility of material surfaces with the characteristic speed $\lambda = 0$ which undergo at most jumps in the transverse component of the first derivative of $\boldsymbol{q}$, the theory accounts for two (real) transverse electromagnetic waves and four coupled thermo-electromagnetic waves whose speeds are real provided only that a certain constitutive condition is required. In **4** we examine symmetric and coupled waves: adopting the same terminology as in magnetofluiddynamics, the fast and slow waves are proven to exist. In **5** we deal with the direct computation of the discontinuity parameters: the electromagnetic waves are no accompanied by any jump in the first derivatives of θ (and also of $\boldsymbol{q}$) and their (purely) electromagnetic properties are closely analogous to the ones of the usual electromagnetic waves in dielectrics, where heat conduction is ignored. Electromagnetic waves turn out to be exceptional [**1**]$_2$, [**11**].

Slow and fast wavefronts are then investigated: the discontinuity vector associated with them is expressed in terms of the parameter characterizing the jump in the first derivative of θ. As it is expected, if we take either $E_t^2 = 0$ or $\varepsilon'(\theta) = 0$ ahead of the wave, the temperature field is uncoupled from the electromagnetic one.

In the latter instance, the fast wave reduces to the usual electromagnetic wave, whereas the slow one yields a temperature rate wave in rigid heat conductor, whose speed is affected by the value of the longitudinal component of $\boldsymbol{q}$ ahead of the wave [**6**], [**15**].

In **6** the growth of one-dimensional coupled waves entering an equilibrium region is investigated using the Thomas technique [3], [19]: the amplitude equation is of Bernoulli type, hence the possibility of shock formation is discussed. It is worth noting that a more general model of electromagneto-thermoelasticity including both thermal relaxation and the electric current density dependent also on $\boldsymbol{g}$ (according to a modified Ohm's law) is presented in [17], in the framework of the linearized equations: employing perturbation techniques, the authors study the influence of small thermoelastic and magnetoelastic couplings on the propagation of plane harmonic waves in unbounded media.

Then a final comment on our results may be at order. In agreement with a well-known general result [20], it is possible to verify that the propagation condition giving all the values for the speed propagation of the wave fronts (within the context of the non-linear basic equations) is identical with the dispersion equation for plane harmonic waves of infinitesimal amplitudes at very high frequencies (within the context of the linearized basic equations).

2 - General formulation and basic equations

Consider a heat conducting electromagnetic rigid body occupying a homogeneous compact region $\mathscr{B}$ of the three dimensional Euclidean space. Let $\boldsymbol{x}$ denote a typical point of $\mathscr{B}$ and $t \in \mathbb{R}^+$ is the time.

For simplicity, the electrical current density $\mathscr{I}$ and the external heat supply are assumed to be absent; we also ignore ferromagnetic bodies.

Thus, in our theory each thermodynamic process consists of a 9-tuples of functions defined on $\mathscr{B} \times \mathbb{R}^+$, whose values have the following physical interpretation: the electric intensity field $\mathbf{E}$, the electric induction $\mathbf{D}$, the magnetic intensity field $\mathbf{H}$, the magnetic induction $\mathbf{B}$, the electric charge density Q, the internal specific energy (per unit volume) $\mathscr{E}$, the specific entropy (per unit volume) η, the absolute temperature $\theta > 0$ and the heat flux $\boldsymbol{q}$.

Throughout this work, the notation is just the customary one in the current literature.

When sufficient smoothness is assumed on the process, Maxwell's equations and the law of balance of energy (i.e. the first law of thermodynamics) take respectively the forms [5], [7]

$$\nabla \times \mathbf{H} = \dot{\mathbf{D}}, \qquad \nabla \cdot \mathbf{D} = Q, \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \qquad \nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$\dot{\mathscr{E}} = -\nabla \cdot \boldsymbol{q} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}, \tag{2.3}$$

an upper dot standing for the partial time derivative.

We assume the second law of thermodynamics to be expressed as

$$\dot{\eta} + \nabla\cdot(\boldsymbol{q}/\theta) = \gamma \geqslant 0 \,, \tag{2.4}$$

i.e. the rate of production of entropy is non negative.

If we introduce the specific free enthalpy density $\zeta = \zeta(\boldsymbol{x}, t)$ through

$$\zeta = \mathscr{E} - \theta\eta - \mathbf{H}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{D} \,, \tag{2.5}$$

then (2.3) and (2.4) yield the Clausius-Duhem inequality in the form

$$-\theta(\dot{\zeta} + \eta\dot{\theta} + \mathbf{B}\cdot\dot{\mathbf{H}} + \mathbf{D}\cdot\dot{\mathbf{E}}) - \boldsymbol{q}\cdot\boldsymbol{g} = \theta^2\gamma \geqslant 0 \,, \quad \text{where } \boldsymbol{g} = \nabla\theta \,. \tag{2.6}$$

In order to account for relaxation phenomena associated with heat conduction, as our constitutive assumptions we suppose that the material, at each point $\boldsymbol{x}$, is characterized by four response functionals $\hat{\zeta}, \hat{\eta}, \hat{\mathbf{D}}, \hat{\mathbf{B}}$ giving the present values of $\zeta, \eta, \mathbf{D}$ and $\mathbf{B}$, whenever $\mathbf{E}, \mathbf{H}\, \theta, \boldsymbol{g}$ and $\boldsymbol{q}$ are known at $(\boldsymbol{x}, t)$, i.e.

$$\zeta = \hat{\zeta}(\mathbf{E}, \mathbf{H}, \theta, \boldsymbol{g}, \boldsymbol{q}) \,, \quad \eta = \hat{\eta}(\mathbf{E}, \mathbf{H}, \theta, \boldsymbol{g}, \boldsymbol{q}) \,, \quad \text{etc.} \tag{2.7}$$

This is in agreement with the point of view of the extended irreversible thermodynamics [**9**] according to which the response functionals may, in non equilibrium, also depend on quantities vanishing in equilibrium; in the present theory the heat flux $\boldsymbol{q}$ behaves as an extra variable.

Hence, we say that our thermodynamic process in $\mathscr{B}\times\mathbb{R}^+$ is admissible if it is compatible with the field equations (2.1), (2.3) complemented by the constitutive relations (2.7); the response functionals are required to satisfy also the dissipation inequality (2.6). We investigate now the thermodynamic restrictions imposed on the afore-mentioned constitutive scheme by (2.6).

By using standard compatibility arguments, it follows that the response functionals $\hat{\zeta}, \hat{\eta}, \hat{\mathbf{B}}$ and $\hat{\mathbf{D}}$ depend on at most $\mathbf{E}, \mathbf{H}, \theta$ and $\boldsymbol{q}$ and further that the response functional $\hat{\zeta}$ must completely determine $\hat{\eta}, \hat{\mathbf{B}}$ and $\hat{\mathbf{D}}$ through the relations [**5**]

$$\eta = \hat{\eta}(\mathbf{E}, \mathbf{H}, \theta, \boldsymbol{q}) = -\hat{\zeta}_\theta \,, \quad \mathbf{B} = -\hat{\zeta}_{\mathbf{H}} \,, \quad \mathbf{D} = -\hat{\zeta}_{\mathbf{E}} \,, \tag{2.8}$$

where, as usual, the subscripts $\theta, \mathbf{H}$ etc. denote partial derivatives. In consistence with the fact that phenomenological statements must be linear in the (extra) variables vanishing in equilibrium, we henceforth suppose that $\zeta_{\boldsymbol{q}} = \alpha(\theta)\boldsymbol{q}$, where $\alpha(\theta)$ is a phenomenological coefficient, e.g. see [**9**]. In view of this hypothesis, what remains of the inequality (2.6) can be written as

$$-\boldsymbol{q}\cdot(\theta\alpha(\theta)\dot{\boldsymbol{q}}+\boldsymbol{g})=\theta^2\gamma\geqslant 0\,. \tag{2.9}$$

The requirement (2.9) is straightway satisfied if $\boldsymbol{q}$ is assumed to meet the relation

$$\boldsymbol{q}=-\chi(\alpha(\theta)\theta\dot{\boldsymbol{q}}+\boldsymbol{g})\,, \tag{2.10}$$

$\chi>0$ being the (constant) coefficient of heat conduction.

In order to specialize the model at hand, we now choose a free enthalpy density function ζ dependent on θ and on the quadratic invariants E^2, H^2, and q^2 in the form

$$\zeta=\hat{\zeta}(\boldsymbol{E},\boldsymbol{H},\theta,\boldsymbol{q})=\zeta^*(\theta)-\frac{1}{2}\left(\varepsilon(\theta)E^2+\mu H^2\right)+\frac{\tau}{2}(\chi\theta)^{-1}q^2\,, \tag{2.11}$$

where $\mu>0$ is the constant magnetic permeability of the dielectric, $\varepsilon(\theta)>0$ plays the role of the dielectric coefficient and $\tau>0$ is the (constant) thermal relaxation time; (2.11) holds within a suitable range of temperatures. The choice (2.11) provides the most direct extension of the classical thermoelectromagnetic theory.

In fact, substitution of (2.11) into (2.8), gives

$$\eta=-\zeta^*_\theta(\theta)+\frac{1}{2}\,\varepsilon'(\theta)E^2+\frac{\tau}{2}(\chi\theta^2)^{-1}q^2\,,\qquad \mathbf{B}=\mu\mathbf{H}\,,\qquad \mathbf{D}=\varepsilon(\theta)\mathbf{E}\,, \tag{2.12}$$

whilst from (2.9) it easily follows the well-known Maxwell-Cattaneo constitutive equation [**2**]

$$\tau\dot{\boldsymbol{q}}=-\boldsymbol{q}-\chi\boldsymbol{g}\,. \tag{2.13}$$

Precisely, when ε is independent of θ and τ is equal to zero, eqs. $(2.12)_{2,3}$ become the usual linear equations of an electromagnetic theory, see e.g. [**5**], [**7**] and (2.13) yields the Fourier's constitutive equation.

Consider now the internal energy $\mathscr{E}$: on account of (2.5), (2.11) and (2.12), it may be expressed as

$$\mathscr{E}=\hat{\mathscr{E}}(\mathbf{E},\mathbf{H},\theta,\boldsymbol{q})=\zeta^*(\theta)-\zeta^*_\theta(\theta)\theta+\frac{1}{2}\left(\varepsilon(\theta)+\theta\varepsilon'(\theta)\right)E^2+\frac{\mu}{2}H^2+\frac{\tau}{\chi\theta}q^2\,, \tag{2.14}$$

whence the specific heat (per unit volume) which is defined by $C = \mathscr{E}_\theta$, will depend on θ, $\mathbf{E}$ and $\boldsymbol{q}$ in the form

$$C = \hat{C}(\mathbf{E}, \theta, \boldsymbol{q}) = C_E(\theta) + C_N(\mathbf{E}, \theta, \boldsymbol{q}) , \tag{2.15}$$

where

$$C_E(\theta) = -\theta\zeta^*_{\theta\theta}(\theta) \tag{2.16}$$

is the usual positive specific heat at equilibrium (where $\tau = 0$ and $\varepsilon'(\theta) = 0$) and

$$C_N(\mathbf{E}, \theta, \boldsymbol{q}) = \frac{1}{2}\left(\varepsilon''(\theta)\theta + 2\varepsilon'(\theta)\right)E^2 - \frac{\tau}{\chi\theta^2}q^2 \tag{2.17}$$

is its non equilibrium value.

As it is customary in thermodynamic theories, we assume that the specific heat is positive, namely $C_E + C_N \geqslant C_m > 0$.

Under the condition

$$C_E + \tfrac{1}{2}\left(\varepsilon''(\theta)\theta + 2\varepsilon'(\theta)\right)E^2 > 0 , \tag{2.18}$$

our requirement is clearly satisfied when a suitable bound for q^2 is introduced. Look now at the balance equations (2.1)-(2.3) complemented by (2.13): on appealing to the constitutive relations $(2.12)_{2,3}$, in view of the expression (2.14) for $\mathscr{E}$, a straightforward calculation allows us to arrive at the system of basic equations governing the model under consideration

$$(2.19)_1 \qquad \nabla\times\mathbf{H} = \varepsilon\dot{\mathbf{E}} + \varepsilon'\dot{\theta}\mathbf{E} , \qquad \varepsilon\nabla\cdot\mathbf{E} + \varepsilon'\mathbf{E}\cdot\boldsymbol{g} = Q ,$$

$$(2.19)_2 \qquad \nabla\times\mathbf{E} = -\mu\dot{\mathbf{H}} , \qquad \nabla\cdot\mathbf{H} = 0 ,$$

$$(2.19)_3 \qquad \left(C_E + \varepsilon''\theta E^2/2 - \tau(\chi\theta^2)^{-1}q^2\right)\dot{\theta} + \varepsilon'\theta\mathbf{E}\cdot\dot{\mathbf{E}} + 2\tau(\chi\theta)^{-1}\boldsymbol{q}\cdot\dot{\boldsymbol{q}} + \nabla\cdot\boldsymbol{q} = 0 ,$$

$$(2.19)_4 \qquad \tau\dot{\boldsymbol{q}} = -\boldsymbol{q} - \chi\boldsymbol{g} .$$

3 - Wave propagation in thermo-electromagnetic solids

In this section, we recall some well-known notions of the theory of non-linear wave propagation.

Let $\Sigma \subset \mathscr{B}\times\mathbb{R}^+$ be a singular surface propagating through the material, represented by an equation of the form $\varphi(\boldsymbol{x}, t) = 0$; $\boldsymbol{n}$ is the unit normal vector to Σ (drawn in the direction of propagation) and λ is the speed of propagation of Σ given by the following formulae

$$\lambda = -\dot{\varphi}/|\nabla\varphi| , \qquad \boldsymbol{n} = \nabla\varphi/_{|\nabla\varphi|} . \tag{3.1}$$

Σ is then said to be a thermo-electromagnetic rate wave if the quantities $\mathbf{E}, \mathbf{H}, \theta$ and $\boldsymbol{q}$ are continuous on $\mathscr{B}\times\mathbb{R}^+$, but their first and second derivatives (with respect to space variables and time t) may suffer jump discontinuities across Σ, while are continuous everywhere else on $(\mathscr{B}\times\mathbb{R}^+)/\Sigma$.

On adopting the standard bracket notation $[G] = G^- - G^+$ to define the jump of an arbitary unknown field G (for instance $\dot{E}$) across the wave, where G^- and G^+ are the values of G immediately behind and ahead of Σ, respectively, let us denote by $\delta f = [(\partial/\partial\varphi)f]$ the discontinuity parameter relative to f which characterizes the jump of the first derivatives of f.

In what follows, we make use of the substitution $[\mathbf{1}]_1$

$$\frac{\partial}{\partial t} f \Rightarrow -\lambda\,\delta f , \qquad \nabla\times\boldsymbol{f}\rightarrow \boldsymbol{n}\wedge\delta\boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{f} \Rightarrow \boldsymbol{n}\cdot\delta\boldsymbol{f}, \qquad \nabla f \Rightarrow \delta f\boldsymbol{n} \tag{3.2}$$

according as f is a scalar function (for instance θ) as a vectorial function. Taking into account (3.2), system (2.19) easily provides the following characteristic equations

$$\boldsymbol{n}\wedge\delta\mathbf{H} = -\varepsilon'\lambda\delta\theta\mathbf{E} - \varepsilon\lambda\delta\mathbf{E} , \qquad \varepsilon\delta E_n + \varepsilon'\delta\theta E_n = 0 , \tag{3.3}$$

$$\boldsymbol{n}\wedge\delta\mathbf{E} = \mu\lambda\delta\mathbf{H} , \qquad \delta H_n = 0 , \tag{3.4}$$

$$(C - \varepsilon' E^2)\lambda\,\delta\theta + \varepsilon'\theta\lambda\mathbf{E}\cdot\delta\mathbf{E} + \frac{2\tau}{\chi\theta}\lambda\boldsymbol{q}\cdot\delta\boldsymbol{q} - \delta q_n = 0 , \tag{3.5}$$

$$\tau\lambda\delta\boldsymbol{q} = \chi\delta\theta\boldsymbol{n} , \tag{3.6}$$

where the subscript $\boldsymbol{n}$ denotes the normal component of an arbitrary vector, i.e. $\delta E_n = \delta\mathbf{E}\cdot\boldsymbol{n}$ etc.

Let us first note that an immediate consequence of the constraint $\nabla\cdot\mathbf{H} = 0$ is $\delta H_n = 0$ (see $(3.4)_2$), namely $\delta\mathbf{H}$ is tangent to the wave front: this result follows also from $(3.4)_1$, if $\lambda\neq 0$.

From (3.6), in the hypothesis $\lambda\neq 0$, we have $\delta\boldsymbol{q} = \delta q_n\boldsymbol{n}$ (longitudinal discontinuity).

To begin with, we investigate whether material surfaces (with $\lambda = 0$) may occur in the present theory.

If $\lambda = 0$, eq. (3.6) gives

$$\delta\theta = 0 , \tag{3.7}$$

while it is satisfied for arbitrary values of $\delta\boldsymbol{q}$.

In view of (3.7), from (3.3), (3.4) and (3.5) it easily follows

$$\delta\mathbf{E} = \delta\mathbf{H} = 0\,, \qquad \delta q_n = 0\,, \qquad \delta \boldsymbol{q}_t \neq 0\,. \tag{3.8}$$

Therefore, we may conclude that material surfaces with the characteristic speed $\lambda = 0$ may exist: they are characterized by (3.7) and (3.8), i.e. only $\delta\boldsymbol{q}_t$ may undergo a jump across them. This means that $\lambda = 0$ has multiplicity 2.

As we are interested in wave fronts, throughout this work we assume $\lambda \neq 0$ and consequently $\delta\boldsymbol{q}_t = 0$.

It is always possible to refer the jump equations (3.3)-(3.6) to a fixed system of cartesian coordinates $(0; x, y, z)$; without any loss of generality we may further select the direction of z-axis in the direction of propagation, i.e. $\boldsymbol{e}_3 = \boldsymbol{n}$.

Let us then denote by the subscripts 1 and 2 the components of $\delta\mathbf{H}$ and $\delta\mathbf{E}$ transverse to $\boldsymbol{n}$ and orthogonal to each other.

We commence by observing that, from $(3.3)_2$ and (3.6), δE_n and δq_n may be written in terms of $\delta\theta$, through the relations

$$\delta E_n = -\frac{\varepsilon'}{\varepsilon} E_n \delta\theta\,, \qquad \delta q_n = \frac{\chi}{\tau\lambda}\,\delta\theta \tag{3.9}$$

and, upon taking the inner product of (3.4) with $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ respectively, we obtain

$$\delta H_1 = -\frac{1}{\lambda\mu}\,\delta E_2\,, \qquad \delta H_2 = \frac{1}{\mu\lambda}\,\delta E_1\,. \tag{3.10}$$

Hence, in view of these results, the jump equations simplify to

$$\begin{gathered}(1-\lambda^2\varepsilon\mu)\,\delta E_1 - \lambda^2\varepsilon'\mu E_1\,\delta\theta = 0\,, \qquad (1-\lambda^2\varepsilon\mu)\,\delta E_2 - \lambda^2\varepsilon'\mu E_2\,\delta\theta = 0\,,\\ \lambda_2\theta\varepsilon' E_1\,\delta E_1 + \lambda^2\theta\varepsilon' E_2\,\delta E_2 + \Big(\lambda^2\Big(C - \varepsilon' E^2 - \frac{\theta\varepsilon'^2}{\varepsilon} E_n^2\Big) + \frac{2}{\theta}\,q_n\lambda - \frac{\chi}{\tau}\Big)\delta\theta = 0.\end{gathered} \tag{3.11}$$

Equations (3.11) represent a linear and homogeneous system in three unknowns δE_1, δE_2, $\delta\theta$: it has non trivial solutions iff its determinant vanishes. By putting equal to zero the determinant of the coefficients of δE_1, δE_2 and $\delta\theta$, the propagation condition follows straightway

$$(1-\lambda^2\varepsilon\mu)\Big\{(\varepsilon\mu\lambda^2-1)\Big(\lambda^2\Big(C-\varepsilon' E^2 - \frac{\theta\varepsilon'^2}{\varepsilon} E_n^2\Big) + \frac{2}{\theta}\,q_n\lambda - \frac{\chi}{\tau}\Big) - \varepsilon'^2\mu E_t^2\lambda^4\Big\} = 0, \tag{3.12}$$

with $E_t^2 = E_1^2 + E_2^2$. From (3.12), it follows directly

$$1 - \varepsilon\mu\lambda^2 = 0 \,, \tag{3.13}$$

$$w(\lambda) = \varepsilon\mu\left(C - \frac{\varepsilon'}{\varepsilon}(\varepsilon + \theta\varepsilon')E^2\right)\lambda^4 + \frac{2}{\theta}\varepsilon\mu q_n\lambda^3 - \left(C - \frac{\varepsilon'}{\varepsilon}(\varepsilon + \theta\varepsilon')E^2 + \varepsilon\mu\frac{\chi}{\tau} + \frac{\theta\varepsilon'^2}{\varepsilon}E_t^2\right)\lambda^2 - \frac{2}{\theta}q_n\lambda + \frac{\chi}{\tau} = 0 \,. \tag{3.14}$$

Let us first examine (3.13); as $\varepsilon(\theta) > 0$, it yields two distinct and real roots given by

$$\lambda = \lambda_e^{\pm} = \pm\,(\varepsilon(\theta)\mu)^{-1/2} \,. \tag{3.15}$$

By analogy with the usual terminology, the corresponding waves are called electromagnetic waves; meanwhile, we note that the expressions for $\lambda_e^{\pm}$ show the influence of the temperature field.

We look now at (3.14); it is easy to see that

$$w(0) = \frac{\chi}{\tau} > 0 \,, \qquad w(\lambda_e^{\pm}) = -\frac{\varepsilon'^2}{\varepsilon}\theta\lambda_e^2 E_t^2 < 0 \tag{3.16}$$

and, provided that the additional restriction

$$C - \frac{\varepsilon'}{\varepsilon}(\varepsilon + \theta\varepsilon')E^2 > 0 \,, \tag{3.17}$$

holds, it follows $w(\lambda) \to +\infty$, in the limits $\lambda \to \pm\infty$.

Therefore, (3.14) should have four distinct and real roots λ_1^-, λ_2^-, λ_3^+, λ_4^+, which verify the relations

$$\lambda_1^- < \lambda_e^- < \lambda_2^- < 0 \,, \qquad 0 < \lambda_3^+ < \lambda_e^+ < \lambda_4^+ \,. \tag{3.18}$$

We may conclude that our system of basic equations (2.19) is not totally hyperbolic; anyhow, without further restrictions on the structure of the material, (3.14) may admit imaginary values for λ. It is noteworthy that, in the particular instance $\varepsilon = \varepsilon_0(\theta + \theta_0)^{-1}$ (ε_0, θ_0 positive constants), (3.17) reduces to the initial requirement $C > 0$.

When the physical structure of the dielectric heat conductor enables us to meet (3.17), the present theory bears evidence of six wavefronts: (i) the electromagnetic wavefronts travelling with the speeds $\lambda_e^{\pm}$; (ii) the slow wavefronts, which are two waves of thermoelectromagnetic type, whose speeds are λ_2^- and $\lambda_3^+(|\lambda_2^-| < \lambda_e^+, \lambda_3^+ < \lambda_e^+)$; (iii) the fast wavefronts whose speeds are the remaining real roots to (3.14), λ_1^-, $\lambda_4^+(|\lambda_1^-| > \lambda_e^+, \lambda_4^+ > \lambda_e^+)$. It is worth examining two particular cases.

Case 1. Let $E_t = 0$, which corresponds to propagation into a region where **E** is parallel to the normal $\boldsymbol{n}$.

In this instance, (3.14) gives again the characteristic speeds $\lambda = \lambda_e^\pm$ and

$$\lambda_{te}^\pm = \Big(C - \frac{\varepsilon'}{\varepsilon}(\varepsilon + \theta\varepsilon')E^2\Big)^{-1}\Big(-\frac{1}{\theta}q_n \pm \sqrt{\frac{q_n^2}{\theta^2} + \frac{\chi}{\tau}\Big(C - \frac{\varepsilon'}{\varepsilon}(\varepsilon + \theta\varepsilon')E^2\Big)}\Big). \tag{3.19}$$

Let us observe that λ_{te}^+ and λ_{te}^- are different in sign and $\lambda_{te}^+ \gtreqless |\lambda_{te}^-|$ according as $q_n \lesseqgtr 0$ ahead of the wave; clearly, they are real values if (3.17) holds.

Case 2. Let $E_t = 0$ and $\varepsilon'(\theta) = 0$ (uncoupled waves). Then the roots (3.19) reduce to the form

$$\lambda_t^\pm = -\frac{q_n}{C'\theta} \pm \sqrt{\frac{q_n^2}{(C'\theta)^2} + \frac{\chi}{\tau C'}}, \tag{3.20}$$

$C' = C_E - (\tau/\chi\theta^2)q^2$ being the positive specific heat. Taking into account that $C' > 0$, λ_t^+ and λ_t^- are real; $\lambda_t^+ \gtreqless |\lambda_t^-|$ according as $q_n \lesseqgtr 0$.

This case is interesting, as it exemplifies the situation when the temperature field is quite uncoupled from the electromagnetic field. In fact $\lambda_t^\pm$ are the speeds of a temperature rate wave in a heat conducting solid (second sound in solids [**6**], [**15**]), and $\lambda_e^\pm$ are the constant speeds of an electromagnetic wave in a dielectric where thermal effects are neglected. Electromagnetic waves are exceptional i.e. they never produce characteristic shocks [**11**]. On the contrary, thermal waves are not exceptional, for instance see [**15**].

4 - Symmetric waves.

A necessary and sufficient condition to obtain symmetric roots from (3.14) is to vanish the coefficients of λ and λ^3 hence, in the present case, the symmetry requirement implies $q_n = 0$ ahead of the wave front.

For convenience, we make use of the definitions

$$\lambda_e^2 = (\varepsilon(\theta)\mu)^{-1}, \qquad \lambda_T^2 = \chi/\tau\,\Big(C - \frac{\varepsilon'}{\varepsilon}(\varepsilon + \theta\varepsilon')E^2\Big), \tag{4.1}$$

provided $C - (\varepsilon'/\varepsilon)(\varepsilon + \theta\varepsilon')E^2 = C^* > 0$.

Substitution of $q_n = 0$ into (3.14), leads to the new wave speed equation which, in a straightforward matter, may be rearranged as

$$(4.2)\qquad (\lambda^2 - \lambda_e^2)(\lambda^2 - \lambda_T^2) = \frac{\varepsilon'^2\theta}{\varepsilon C^*} E_t^2 \lambda_e^2 \lambda^2 .$$

A direct calculation shows that (4.2) has distinct and real roots given by

$$(4.3)\qquad \lambda_{f,s}^{\pm} = \pm \{2^{-1}(\lambda_e^2 (1 + \frac{\varepsilon'^2\theta}{\varepsilon C^*} E_t^2) + \lambda_T^2 \pm \sqrt{(\lambda_e^2 (1 + \frac{\varepsilon'^2\theta}{\varepsilon C^*} E_t^2) - \lambda_T^2)^2 + 4\lambda_e^2 \lambda_T^2 \frac{\varepsilon'^2\theta}{\varepsilon C^*} E^2}\}^{1/2},$$

moreover, the following inequality holds

$$(4.4)\qquad \lambda_s^2 \leqslant (\lambda_e^2, \lambda_T^2) \leqslant \lambda_f^2$$

the equality sign occurring if either $E_t = 0$ or $\varepsilon'(\theta) = 0$; in the latter case $\lambda_f^{\pm} = \lambda_e^{\pm}$ and $\lambda_s^{\pm} = \pm (\chi/\tau C')^{\frac{1}{2}}$, i.e. the fast wavefronts are the electromagnetic waves in the dielectric and the slow wavefronts become symmetric temperature rate waves [**6**], [**12**].

In conclusion, provided only that (3.17) holds, eq. (4.2) gives two real and positive values for λ^2, hence we have two possible wave fronts of thermo-electromagnetic type in the forward direction: the fast wavefront travelling with speed λ_f^+ greater that λ_e^+ and λ_T^+, the slow wavefront whose speed λ_s^+ is smaller than λ_e^+ and λ_T^+; propagation in the opposite direction with equal speed is also possible.

The limit $E_t \to 0$ ahead of the wave, provides the customary electromagnetic wave in the dielectric (fast wave), whereas the slow wavefront tend to a temperature wave whose speed λ_T^+ is affected by the coupling coefficient $\varepsilon(\theta)$ and $\lambda_T^+ \gtreqless (\chi/C_E\tau)^{\frac{1}{2}}$ according as $(1/2\varepsilon)(\varepsilon''\varepsilon - 2\varepsilon'^2)\theta E^2 - (\tau/\chi\theta^2)q^2 \gtreqless 0$, where $(\chi/C_E\tau)^{\frac{1}{2}}$ is the so called second sound speed which is $v_s/\sqrt{3}$, if v_s denotes the speed of the first sound, see e.g. [**4**].

5 - Calculation of the discontinuity parameters

(i) *Electromagnetic waves.*

As it should be expected, the necessary condition for the existence of wavefronts propagating with the speed $\lambda_e^{\pm} = \pm (\varepsilon(\theta)\mu)^{-1/2}$ is that $\delta\theta = 0$ (and also $\delta q_n = 0$) and the transverse discontinuity vectors $\delta\mathbf{H}$ and $\delta\mathbf{E}(\delta E_n = \delta H_n = 0)$ are mutually orthogonal, this condition being also sufficient as soon as $\delta\mathbf{H}\cdot\mathbf{E} \neq 0$.

On account of (3.3) and (3.4), we have

$$(1 - \varepsilon\mu\lambda^2)\,\delta\mathbf{H} = \varepsilon'\,\delta\theta\lambda\boldsymbol{n}\wedge\mathbf{E}\,, \tag{5.1}$$

and upon taking the inner product of (5.1) with **E**, the sufficient condition follows immediately.

On the other hand, if $\lambda = \lambda_e^{\pm}$, relations (3.10) become

$$\delta E_2 = \mp Z\delta H_1\,, \qquad \delta E_1 = \pm Z\delta H_2\,, \tag{5.2}$$

where $Z = (\mu/\varepsilon)^{\frac{1}{2}}$ is the impedence of the dielectric.

In view of $\delta H_n = 0$, (5.2) yields $\delta\mathbf{E}\cdot\delta\mathbf{H} = 0$.

If $E_t^2 \neq 0$ ahead of the wave, $(3.11)_{1,2}$ imply $\delta\theta = 0$, $\delta\mathbf{E}_t \neq 0$, hence from (3.9) we have

$$\delta E_n = \delta q_n = 0\,. \tag{5.3}$$

Look now at (3.5): on appealing to the previous results, it follows

$$\mathbf{E}\cdot\delta\mathbf{E} = 0\,. \tag{5.4}$$

This means that the absolute value of the electric field remains constant and that the electric field rotates across the wave front.

Hence, our theory exhibits homothermal transverse electromagnetic waves, whose peculiar properties are strictly analogous to the ones of the ordinary electromagnetic waves in dielectric solids, without heat conduction, whenever $\varepsilon(\theta)$ is viewed as the dielectric coefficient, see e.g. $[\mathbf{18}]_{1,2}$.

Another noteworthy property of the electromagnetic waves is that they are exceptional.

As shown above, for any characteristic value λ_e^+ and λ_e^-, we find two linearly independent discontinuity vectors, respectively given by

$$\delta\Pi_1^{\pm} = (\delta H_n,\, \delta H_1,\, \delta H_2,\, \delta E_n,\, \delta E_1,\, \delta E_2,\, \delta\theta,\, \delta q_n) = (0,\, 0,\, \pm Z^{-1}\delta E_1,\, 0,\, \delta E_1,\, 0,\, 0,\, 0),$$

$$\delta\Pi_2^{\pm} = (0,\, \mp Z^{-1}\delta E_2,\, 0,\, 0,\, 0,\, \delta E_2,\, 0,\, 0)\,.$$

Taking account that $\lambda_e^{\pm} = \pm\,(\varepsilon(\theta)\mu)^{-1/2}$ depend on θ alone, it is a simple matter to prove that Lax' condition of exceptionality holds i.e. $\delta\lambda = 0$ (see [**11**], $[\mathbf{1}]_2$). As a final comment, let us remark that in the limiting case $\tau \to 0$ (when $(2.19)_4$ reduces to the parabolic Fourier's law), the propagation condition associated with (2.19) yields $\lambda^2 = (\varepsilon(\theta)\mu)^{-1}$; consequently also the limit $\tau \to 0$ accounts for the existence of transverse electromagnetic waves ($\delta E_n = \delta H_n = 0$) which are homothermal ($\delta\theta = 0$) and further $\delta \mathbf{E}\cdot\delta\mathbf{H} = 0$, but, instead of (5.4), we find $\delta q_n = \varepsilon'\theta\lambda\mathbf{E}\cdot\delta\mathbf{E}$, while $\delta \boldsymbol{q}_t$ is now undetermined.

(ii) *Thermo-electromagnetic waves. Slow and fast symmetric waves.*

Let us assume that λ is neither equal to zero nor to $\lambda_e^{\pm}$.

Cumbersome calculations allow us to arrive at the explicit expression of the discontinuity vector associated with the waves of thermo-electromagnetic type, in terms of the parameter characterizing the discontinuities of the first derivatives of θ.

From equations (3.9), (3.10) and (3.11), we obtain

$$\delta\Pi = \Big(0, \varepsilon' E_2\lambda_e^2\lambda, -\varepsilon' E_1\lambda_e^2\lambda, -\frac{\varepsilon'}{\varepsilon}E_n(\lambda^2-\lambda_l^2), -\frac{\lambda^2\varepsilon'}{\varepsilon}E_1, -\frac{\lambda^2\varepsilon'}{\varepsilon}E_2, \lambda^2-\lambda_e^2, \frac{\chi}{\tau\lambda}(\lambda^2-\lambda_e^2)\Big)\cdot\frac{1}{(\lambda^2-\lambda_e^2)}\,\delta\theta\,, \tag{5.5}$$

where λ takes either the values $\lambda_f^{\pm}$ or $\lambda_s^{\pm}$.

Let us observe that (5.5) is just the same solution as the non symmetric case, when $q_n \neq 0$, provided that λ is given by a real solution of the general equation (3.14).

As it usually happens, the waves of thermo-electromagnetic type are not generally exceptional, i.e. it is possible to determine a finite critical time at which their amplitude growths indefinitely. So these waves can develop into shoch waves.

We shall now proceed and give details of the growth or decay of the amplitude of one-dimensional coupled waves polarized in the (x, y) plane and propagating along the z-direction.

6 - Special case: propagation of one-dimensional thermo-electromagnetic waves entering a region at equilibrium and amplitude equation

In this section, we confine our attention to the propagation of coupled plane waves along the z-axis; henceforth, all the field quantities depend on z and t alone, precisely we suppose $\mathbf{E} = E(z,t)\boldsymbol{e}_1$, $\mathbf{H} = H(z,t)\boldsymbol{e}_2$, $\theta(z,t)$ and $\boldsymbol{q} = q(z,t)\boldsymbol{n}$.

In order to simplify the ensuing calculations, it is also assumed that the region ahead of the wave has always been at a constant state, both thermal and electromagnetic, then $\dot{E}^+(z,t) = \dot{H}^+(z,t) = \dot{\theta}^+(z,t) = 0$ and also $q^+(z,t) = 0$ henceforth, the subscript « 0 » shall indicate that the quantities are evaluated at equilibrium, i.e. at the state $\Pi_0 = (E_0, H_0, \theta_0, 0)$.

The basic equations (2.19) read now as follows

$$
(6.1)\qquad \begin{aligned} &-H_{,z}=\varepsilon\dot{E}+\varepsilon' E\dot{\theta}\,, && E_{,z}=-\mu\dot{H}\,,\\ &(C-\varepsilon' E^2)\dot{\theta}+\varepsilon'\theta E\dot{E}+\frac{2\tau}{\chi\theta}q\dot{q}+q_{,z}=0, && \tau\dot{q}=-q-\chi\theta_{,z}\,, \end{aligned}
$$

where a comma followed by the index z, denotes partial differentiation with respect to z.

Throughout this section, it is assumed $\lambda > 0$.

For definitess, let us rewrite some details of the one-dimensional counterpart of our previous results (see **3**).

Precisely, the trajectory Σ is said to be a one-dimensional thermo-electromagnetic wavefront if the fields $E(z,t)$, $H(z,t)$, $\theta(z,t)$ and $q(z,t)$ are jointly continuous in z, t on $\mathscr{B}\times\mathbb{R}^+$, while their first and second derivatives (in both variables z and t) have at most a jump discontinuity across Σ, being continuous everywhere else.

On the basis of our assumptions, we have $[\dot{E}] = \dot{E}^-$ etc., however for the sake of convenience, here and in what follows, we use the notation $\dot{E}, \dot{H}, \dot{\theta}$ instead of $\dot{E}^-$ etc.

Further, when we form the jump of each term in the equations (6.1), all subsequent (continuous) coefficients are evaluated ahead of the wave, i.e. in the constant state Π_0.

Let us suppose $[\dot{\theta}] \neq 0$, consequently we here exclude the one-dimensional electromagnetic waves, propagating with the constant speed λ_{e0}.

Clearly, the derivation of the equation governing wavespeeds is just a standard application of the conditions of compatibility given in e.g. [**21**]: if f is continuous at the wave, it follows that $[f_{,z}] = -\lambda^{-1}[\dot{f}]$.

Define

$$
(6.2)\qquad \lambda_{e0}^2=(\varepsilon\mu)_0^{-1}\,,\qquad \lambda_{T0}^2=\chi/\tau\,\Big(C-\frac{\varepsilon'}{\varepsilon}(\varepsilon+\theta\varepsilon')E^2\Big)_0,
$$

then the propagation condition is closely analogous to (4.2), provided the addition of the subscript « 0 » to all coefficients therein.

Under the requirement $(C_E + (1/2\varepsilon)(\varepsilon\varepsilon''-\varepsilon'^2)\theta E^2)_0 > 0$, the propagation condition yields two real (and constant) solutions for λ, λ_{f0}, λ_{s0} characterized by the inequality $\lambda_{s0} < (\lambda_{e,0}, \lambda_{T0}) < \lambda_{f0}$, then there exist two modes of propagation specified by the speeds λ_{f0} (fast wave) and λ_{s0} (slow wave).

Let us investigate the fast wave (obviously, similar considerations hold for the slow one) propagating with the constant speed λ_{f0}. Unless stated otherwise, we henceforth omit the « 0 » notation.

The discontinuity vector associated with them, reduces to

$$\delta\Pi = (\dot{H}, \dot{E}, \dot{\theta}, \dot{q}) = \left(\varepsilon' E\lambda_e^2\lambda, \frac{\varepsilon'}{\varepsilon} E\lambda^2, \; \lambda_e^2 - \lambda^2, \frac{\chi}{\tau\lambda} \; (\lambda_e^2 - \lambda^2)\right) \frac{a(t)}{\lambda_e^2 - \lambda^2} \tag{6.3}$$

with $\lambda = \lambda_f$ and $a(t) = \dot{\theta}$.

In order to obtain the differential equation giving the time rate of change of the amplitude $a(t)$, we here employ the well-known method of Thomas [19], which is based on the use of the kinematical compatibility conditions on the propagating trajectory $\Sigma(t)$.

Furthermore, let us introduce the usual displacement derivative D_t (or Thomas derivative) expressed by [21]

$$D_t[\dot{G}] = [\ddot{G}] + \lambda[\dot{G}_{,z}] \, . \tag{6.4}$$

If A and B are generic quantities that suffer a jump discontinuity across Σ, then

$$[AB] = A_0[B] + B_0[A] + [A][B] = AB \, . \tag{6.5}$$

Indeed the differential equation governing the evolutionary behaviour of $a(t)$ is derived by following a fairly standard procedure, but for completeness we present here a brief sketch of the proof. Let us first differentiate equations (6.1) with respect to time t, then we proceed and take the jumps of the resulting expressions across Σ.

With the aid of (6.4) and (6.5), the induced discontinuities $\dot{H}_{,z}$ and $\dot{q}_{,z}$ are immediately eliminated. On using formula (6.3), in view of the fact that $D_t\lambda = 0$, other straightforward calculations enable us to arrive at the following equations in $\dot{E}_{,z}$ and $\dot{\theta}_{,z}$

$$(\lambda^2 - \lambda_e^2)\dot{E}_{,z} + \frac{\varepsilon'}{\varepsilon} E\lambda^2\dot{\theta}_{,z} + \frac{2\varepsilon' E\lambda\lambda_e^2}{\varepsilon(\lambda^2 - \lambda_e^2)} D_t\dot{\theta} - \left(\frac{(\varepsilon''\varepsilon - 2\varepsilon'^2)\lambda^2 - \varepsilon''\varepsilon\lambda_e^2}{\varepsilon^2(\lambda^2 - \lambda_e^2)}\right) E\lambda\dot{\theta}^2 = 0, \tag{6.6}$$

$$\begin{aligned} &\left(\frac{\chi}{\tau} - \left(C_E + \frac{\theta\varepsilon'' E^2}{2}\right)\lambda^2\right)\lambda\dot{\theta}_{,z} - \theta\varepsilon' E\lambda^3\dot{E}_{,z} \\ &+ \left\{\left(C_E + \frac{\theta\varepsilon'' E^2}{2}\right)\lambda^2 + \frac{\chi}{\tau} - \frac{\varepsilon'^2\theta E^2\lambda^4}{\varepsilon(\lambda^2 - \lambda_e^2)}\right\} D_t\dot{\theta} \\ &+ \frac{\chi}{\tau^2}\dot{\theta} + \left\{\frac{2\chi}{\tau\theta} - \frac{\varepsilon'}{\varepsilon}(2\theta\varepsilon'' + \varepsilon')E^2\lambda^4(\lambda^2 - \lambda_e^2)^{-1} + \frac{\theta\varepsilon'^3}{\varepsilon^2} E^2\lambda^6(\lambda^2 - \lambda_e^2)^{-2} \right. \\ &\left. + \tfrac{1}{2}(\varepsilon'' + \theta\varepsilon''')E^2\lambda^2\right\}\dot{\theta}^2 = 0 \, . \end{aligned} \tag{6.7}$$

Equations (6.6), (6.7) are combined in order that the terms in $\dot{E}_{,z}$ and $\dot{\theta}_{,z}$ vanish. To this aim, look at the equation provided by the calculation: $\theta\varepsilon' E\lambda^3$ times (6.6) $+ (\lambda^2 - \lambda_e^2)$ times (6.7).

On account of the propagation condition

$$(6.8)\qquad (\lambda^2 - \lambda_e^2)\left(\lambda^2\left(C_E + \frac{\theta\varepsilon'' E^2}{2}\right) - \frac{\chi}{\tau}\right) = \frac{\theta\varepsilon'^2}{\varepsilon} E^2 \lambda^4 ,$$

the coefficient of $\dot{\theta}_{,z}$ becomes null, so after some cumbersome calculations, we easily establish the desired result.

Amplitude equation. The thermal amplitude $a(t)$ of a one-dimensional wave of thermo-electromagnetic type (fast wave) entering a region at the equilibrium state Π_0, obeys the Bernoulli equation

$$(6.9)\qquad D_t a + \alpha a + \beta a^2 = 0 ,$$

where the constant coefficients α and β are given by

$$(6.10)_1\quad \left\{2\left(\frac{\chi}{\tau}(\lambda^2 - \lambda_e^2)\right) + \lambda_e^2\left(\left(C_E + \frac{\theta\varepsilon'' E^2}{2}\right)\lambda^2 - \frac{\chi}{\tau}\right)\right\}\alpha = \frac{\chi}{\tau^2}(\lambda^2 - \lambda_e^2) ,$$

$$(6.10)_2\quad \left\{2\left(\frac{\chi}{\tau}(\lambda^2 - \lambda_e^2)\right) + \lambda_e^2\left(\left(C_E + \frac{\theta\varepsilon'' E^2}{2}\right)\lambda^2 - \frac{\chi}{\tau}\right)\right\}\beta$$

$$= \left(\frac{2\chi}{\tau\theta} + \left(C_E + (\varepsilon'' + \theta\varepsilon''')\frac{E^2}{2}\right)\lambda^2\right)\cdot(\lambda^2 - \lambda_e^2) - (\varepsilon' + 3\theta\varepsilon'')\frac{\varepsilon'}{\varepsilon}E^2\lambda^4$$

$$+ 3\theta\frac{\varepsilon'^3}{\varepsilon^2}E^2\lambda^6(\lambda^2 - \lambda_e^2)^{-1}) ,$$

with $\lambda = \lambda_f$, $\lambda_f^2 > (\lambda_e^2, \chi/\tau(C_E + \theta\varepsilon'' E^2/2))$.

If we denote by $a(0)$ the initial amplitude, the solution to (6.9) has the usual form

$$(6.11)\qquad a(t) = \frac{1}{\left(\frac{1}{a(0)} + \frac{\beta}{\alpha}\right)\exp\alpha t - \frac{\beta}{\alpha}} .$$

Indeed, the evolutionary behaviour of $a(t)$ is well-known, see e.g. [**3**], however, setting aside a detailed and unuseful investigation of (6.11), some information on the signs of α and β may be at order.

Meanwhile, (6.7) generally accounts for the existence of a finite critical time $t_c > 0$, expressed by the formula

$$t_c = \frac{1}{\alpha} \log \left(\frac{\beta a(0)}{\alpha + \beta a(0)} \right) \tag{6.12}$$

such that in the $\lim t \to t_c$, $|a(t)| \to \infty$; this means that a shock begins to form.

In agreement with physical expectations, α is here positive: this is a direct consequence of (6.8). Further, note that α is also positive for the slow wave front.

On the other hand, t_c exists iff $-\alpha/(\alpha + \beta a(0)) > 0$, which means that β and $a(0)$ must be different in sign, precisely: $a(0) \lessgtr -(\alpha/\beta)(\lessgtr 0)$ according as $\beta \gtrless 0$. When $a(0) = -\alpha/\beta$, (6.11) yields $a(t) = a(0)$ $\forall t \geqslant 0$. The value $-\alpha/\beta$ is usually called the critical amplitude of the fast wave. It may be interesting to appeand some additional comment on the effective possibility of observing shock effects. It is worth pointing out that t_c depends crucially on the order of magnitude of the relaxation time τ. Set $\lambda_T^{*2} = \chi/\tau(C_E + \theta\varepsilon'' E^2/2)$, $\varepsilon'' > 0$, α may be rewritten as

$$\alpha = \frac{\tau^{-1}\lambda_T^{*2}(\lambda_f^2 - \lambda_e^2)}{2(\lambda_T^{*2}(\lambda_f^2 - \lambda_e^2) + \lambda_e^2(\lambda_f^2 - \lambda_T^{*2}))},$$

that is α is proportional to τ^{-1} then, as it seems reasonable to expect the order of magnitude of τ very small (see [**4**] for an estimate of τ, valid for most metals) large initial amplitude occurs for $|a(t)|$ to growth. This means that our wave may evolve into a shock wave in a short time.

Note that a simple case of (6.9) is provided by a heat conducting solid when the coupling coefficient ε is constant: in this instance, $\lambda_s^2 = \chi/\tau C_E$, that is just the usual (constant) square speed of a temperature rate wave in a solid. It is easy to show that the discontinuity vector is $\delta\Pi = (0, 0, 1, \chi/\tau\lambda_s)\dot{\theta}$ and further routine calculations yield $\alpha = 1/2\tau$, $\beta = -1/\theta$, then $-\alpha/\beta = \theta/2\tau > 0$. In conclusion, provided that $a(0) > \tau^{-1}(\theta/2)$, there exists a critical time $t_c = -2\tau \log\,(1 - \theta/2\tau a(0))$ such that, as $t \to t_c$, $|\dot{\theta}(t)| \to \infty$. This result is clearly a direct consequence of the ones stated in [**15**], where a more general hyperbolic system is considered, so as to account for second sound effect in solids.

Let us then suppose to include also the effects of the current density vector $\mathscr{J}$ in our analysis.

In addition to the constitutive relations (2.12), thermodynamics implies also the classical Ohm's law $\mathscr{J} = \sigma\mathbf{E}$, where $\sigma > 0$ is the constant electrical conductivity.

Of course the presence of $\mathscr{I}$, inside the basic equation $(2.19)_{1,3}$ due to the continuity of **E** across Σ, do not change the possible speeds of propagation, listed in **3** and is perhaps remarkable that even the conclusions previously reached on the evolutionary behaviour of $a(t)$ do not seem to be seriously affected.

Remark. It is worth observing that a more general model of thermo-electromagnetism accounting for the current density $\mathscr{I}$ (of course of some interest from a theoretical point of view) may be presented, by introducing another extra variable ($\mathscr{I}$) in the fundamental free enthalpy density relation (2.11) so that, besides the generalized Fourier's law (2.13), from the reduced dissipation inequality a modified Ohm's law of the type: $\tau_1 \dot{\mathscr{I}} + \mathscr{I} = \sigma E$, $\tau_1 (> 0)$ being the electric relaxation time, is just recovered, see e.g. [**12**].

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Summary

In this paper we investigate in detail the non-linear wave propagation in heat conducting dielectric solids within the context of extended linear irreversible thermodynamics. The growth of the discontinuities associated with a one-dimensional thermo-electromagnetic wave propagating into a region at equilibrium and the evaluation of the critical time are finally exhibited.

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