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Generalized chromatic numbers of some graphs (II) (**)

1 – Although this paper is a continuation of [**5**], we make it self-contained by recalling the main definitions and results.

We will adopt the definition in [**1**]$_1$. Let A be a set of natural numbers. The A-chromatic number of a graph G, $g_A(G)$ ($A \subset N$), is the smallest number of colours needed to colour the vertices of G so that the distance between any two vertices with the same colour is not in A.

In [**5**] we determined the chromatic numbers of some graphs when $A = \{1, 3, ...\}$, $A = \{2, 4, ...\}$, or $A = \{c\}$. We will now determine some other results. Let $g'(G)$, $g''(G)$ and $g_c(G)$ denote these numbers, respectively. We call the corresponding colourings odd-colourings, even-colourings and c-colourings, respectively.

As usual, let $\lfloor x \rfloor$ denote the greatest integer not exceeding the number x and $\lceil x \rceil$ the smallest integer not less than x.

2 – Consider now a n-cycle C_n. Let the vertices of C_n be $v_0, v_1, ..., v_{n-1}$. In [**5**] $g_c(C_n)$ has been calculated. We now determine $g'(C_n)$ and $g''(C_n)$.

We first of all prove the following

Lemma. *If n is odd, all vertices with a given colour must lie in one half circle* (1).

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(1) A half circle is a set of vertices labeled $v_i, v_{i+1}, ..., v_{i+\lfloor n/2 \rfloor}$.

Proof. Without loss of the generality let the given colour be 0 and vertex v_1 be coloured with colour 0. Let v' the vertex furthest from v_1 in the clockwise direction which is coloured with the colour 0 and v'' the furthest from v_1 in the counter-clockwise direction which is coloured with colour 0. If v_1, v' and v'' do not lie in one half circle, the shortest path from v' and v'' does not pass through v_1 and thus must be of odd length if n is odd.

Theorem 1. *$g'(C_n) = 2$ if n is even; $g'(C_n) = 3$ if $n = 3, 5, 9$; $g'(C_n) = 4$ otherwise.*

Proof. Consider first of all n even. In this case C_n is bipartite and the result follows readily.

Now let n be odd. In view of the lemma we see that the most vertices can be coloured with any given colour is every other vertex of one half circle. If $n = 2k + 1$, this number is $\lfloor (k+2)/2 \rfloor$. Thus with three colours we can colour at most $3(\lfloor (k+2)/2 \rfloor)$ vertices. If we are to colour all the vertices of C_n with three colours we must have $3(\lfloor (k+2)/2 \rfloor) \geqslant 2k+1$. This inequality holds for $k = 1, 2, 4$ or $n = 3, 5, 9$. In all these case we have colourings using three colours. Namely, to obtain an odd-colouring of C_3 we assign distinct colours to every vertex, to obtain an odd-colouring of C_5 we assign the colour 0 to vertices v_i, v_{i+2}, the colour 1 to vertices v_{i+1}, v_{i+3} and the colour 2 to vertex v_{i+4}. At last to obtain an odd-colouring of C_9 we assign the colour 0 to vertices v_i, v_{i+2}, v_{i+4}, the colour 1 to vertices v_{i-3}, v_{i-1}, v_{i+1}, and the colour 2 to vertices v_{i+3}, v_{i+5}, v_{i+7}. For other odd cycles we can construct a colouring with four colours in the following way. One semicircle is coloured alternately with colours 0 and 1 while the other half is coloured alternately with colours 2 and 3.

Theorem 2. *$g''(C_n) = \lceil n/3 \rceil$ if n is odd, $n \neq 5$; $g''(C_n) = \lceil n/2 \rceil$ otherwise.*

Proof. First we consider n even. In this case we can colour with any given colour at most two vertices. Indeed let v_i, v_j, v_k be three vertices such that, for example, distance between v_i and v_j, $d(v_i, v_j)$, and the distance between v_j and v_k, $d(v_j, v_k)$, are odd. Without loss of the generality suppose $i < j < k$. Then the distance between v_i and v_k is equal to $d(v_i, v_j) + d(v_j, v_k)$ or equal to $n - [d(v_i, v_j) + d(v_j, v_k)]$. So that it is even. A colouring using $n/2$ colours is obtained by colouring v_{2i} and v_{2i+1} with colour i, for $i = 0, \ldots, n/2$.

For n odd we can colour at most three vertices with the same colour. Thus $g''(C_n) \geqslant \lceil n/3 \rceil$. Now consider n odd, $n > 5$. Let $n = 2k+1$ and $j = \lfloor (k+1)/3 \rfloor$.

If $k = 3j + 1$ then $n/3 = 2j + 1$ and with colour i we colour the vertices v_i, $v_{n/3+i}$ and $v_{2n/3+i}$, for $i = 0, \ldots, n/3 - 1$. In this case $g''(C_n) = \lceil n/3 \rceil$.

If $k = 3j$ then $\lceil n/3 \rceil = 2j + 1$ and with colour i we colour the vertices v_{2i}, v_{2j+i+1}, v_{4j+i} for $i = 0, \ldots, 2j - 2$. The uncoloured vertices v_{2j-1}, v_{2j}, v_{6j-1} and v_{6j} can be coloured with two new colours, thus giving a $2j + 1$ colouring. So in this case $g''(C_n) = \lceil n/3 \rceil$.

If $k = 3j - 1$ then $\lceil n/3 \rceil = 2j$ and with colour i we colour vertices v_i, v_{2j-1+i} and v_{4j+i} for $i = 0, \ldots, 2j - 2$.

The vertices v_{4j-2} and v_{4j-1} are left uncoloured, but can be coloured with one more colour, again yielding an $\lceil n/3 \rceil$ colouring.

For $n = 3$ we have the optimal colouring when all vertices receive the colour 0. Thus $g''(C_3) = 1 = \lceil 3/3 \rceil$.

For $n = 5$ it is readily seen that 3 colours must be used, thus $g''(C_5) = \lceil 5/2 \rceil$.

3 – Consider now the complete multipartite graphs. These graphs are not complicated to colour partly because their distance set $D = \{1, 2\}$ is so simple. Thus for instance $g''(K_{n_1, \ldots, n_k}) = g_2(K_{n_1, \ldots, n_k})$ and $g'(K_{n_1, \ldots, n_k}) = g_1(K_{n_1, \ldots, n_k})$.

Theorem 3. *$g'(K_{n_1, \ldots, n_k}) = k$ and $g''(K_{n_1, \ldots, n_k}) = \max\{n_1, \ldots, n_k\}$.*

Proof. Two vertices in the same partition have a distance of two. This means that in the case of an even-colouring all the vertices in any given partition must be differently coloured. We can however use the same colours in different partitions. Thus $g''(K_{n_1, \ldots, n_k}) = \max\{n_1, \ldots, n_k\}$.

For an odd-colouring we must colour vertices in different partitions with different colours. Thus we need at least k colours. However all the vertices of any given partition can be coloured with the same colour, thus yielding a k-colouring.

Next we look at the problem of colouring a special derived graph $G(m)$ when we know the generalized chromatic number of the graph G. $G(m)$ is obtained by taking m copies of G and for each edge in the graph G taking a complete bipartite graph on the corresponding vertices of $G(m)$. In other words $G(m)$ is the lexicografical product of G by $\bar{K}_m$.

Theorem 4. *Let G be a connected graph. Then $g_A(G(m)) = g_A(G)$ if $2 \notin A$; $g_A(G(m)) = m g_A(G)$ if $2 \in A$.*

Proof. This result follows readily from the next relation

$$d_{G(m)}((v, i), (u, j)) = \begin{cases} d_G(v, u) & \text{if } u \neq v \\ 2 & \text{if } u = v, \end{cases}$$

where with (v, j) we denote the j-th copy of the vertex v of G and with $d_G(u, v)$ the distance between u and v in the graph G.

Finally we consider generalized colourings of products of graphs.

Theorem 5. *Let G and H be arbitrary graphs and $L = G \times H$ their cartesian product. Then*

$$\max\{g_A(G), g_A(H)\} \leqslant g_A(L) \leqslant \min\{|V(H)|g_A(G), |V(G)|g_A(H)\} .$$

Proof. As G and H are induced subgraph of L we clearly have $g_A(G) \leqslant g_A(L)$ and $g_A(H) \leqslant g_A(L)$ and the left inequality follows.

The product L can be regarded as formed by $|V(H)|$ induced copies of G with additional edged joining vertices in distinct copies. Therefore a proper A-colouring results if we take $g_A(G)$ colours for each copy of G and use disjoint sets of colours for different copies. This yields an A-colouring with $|V(H)|g_A(G)$ colours. As the role of G and H is symmetric we obtain the right inequality.

Note that in the case of ordinary colourings the left inequality becomes an equality [**6**]. For $G = C_4$, $H = K_2$ and $A = \{2\}$, we get $2 \leqslant g_2(C_4 \times K_2) \leqslant 4$. As $g_2(C_4 \times K_2) = 4$ we see that the right inequality is also possible.

It is interesting that in the next theorem we can prove a closer lower bound for the strong product, a denser graph than the cartesian product.

Recall that in the strong product $G \bullet H$ two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if: (a) $u_1 = u_2$ and v_1 is adjacent to v_2 in H, (b) u_1 is adjacent to u_2 in G and $v_1 = v_2$, (c) u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H.

Theorem 6. *Let G and H be connected graphs and $L = G \bullet H$ their strong product. Then*

$$\max\{g_A(G), g_A(H)\} \leqslant g_A(L) \leqslant g_A(G) \cdot g_A(H) .$$

Proof. As in Theorem 5 the left inequality follows from the fact that our graphs G and H are induced graphs of L. To prove the right inequality consider the following expression for distance in L

$$d_L((u_1, v_1), (u_2, v_2)) = \max\{d_G(u_1, u_2), d_H(v_1, v_2)\} .$$

Thus if X_G and X_H denote two optimal A-colourings of G and H respectively, we can A-colour L by the function $f(u, v) = (X_G(u), X_H(v))$.

This gives the right inequality.

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Abstract

Let A be a set of natural numbers. The A-chromatic number of a graph G, $g_A(G)$, is the smallest number of the colours to colour the vertices of G so that the distance between any two vertices with the colour is not in A. Here the A-chromatic number is determined when A is the set of even integers or the set of odd integers, and G is a cycle or a multipartite graph. Other results are given.

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