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D. P. GUPTA and S. M. MAZHAR (*)

On the degree of approximation by Fourier-Laguerre expansions (**)

1 - Let f be a function defined and Lebesgue-measurable in the interval $[0, \infty)$ such that for $\alpha > -1$, the integrals

$$\int_0^\infty \exp(-u)u^\alpha f(u)\,du\,, \qquad \int_0^\infty \exp(-u)u^\alpha f(u)L_n^{(\alpha)}(u)\,du\,, \qquad n \geqslant 1$$

exist, where $L_n^{(\alpha)}(x)$ denote the Laguerre polynomials of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^\infty L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1}\exp\left(-\frac{x\omega}{1-\omega}\right).$$

The Fourier-Laguerre expansion associated with the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^\infty a_n L_n^{(\alpha)}(x)\,, \tag{1.1}$$

where $\qquad \Gamma(\alpha+1)\binom{n+\alpha}{n} a_n = \int_0^\infty \exp(-x)x^\alpha f(x)L_n^{(\alpha)}(x)\,dx\,.$

Kogbetliantz $[\mathbf{2}]_{1,2,3}$ Szegö $[\mathbf{3}]_{1,2}$ and Gupta $[\mathbf{1}]_1$ discussed the Cesàro sum-

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(*) Indirizzo degli AA.: D. P. GUPTA, B.S. Univ. of Technology, Ajdabia (Libya); S. M. MAZHAR, Dept. of Mathematics, Kuwait University, P.O. Box 5969, Kuwait.

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mability of the series (1.1). Also Gupta [1]$_2$ showed that if

$$\int_t^\delta \frac{|f(u)|}{u}\,du = o\left(\log\frac{1}{t}\right), \qquad \int_1^\infty \exp(-t/2)\,t^{\alpha-k-1/3}|f(t)|\,dt < \infty,$$

then

$$\sigma_n^{(k)}(0) = o(\log n), \qquad k > \alpha + \tfrac{1}{2},$$

where $\sigma_n^{(k)}(0)$ denote the Cesàro means of the series (1.1) at the end point $x = 0$.

The above result covers the case $k = 0$ in the Cesàro scale of summation if $\alpha + \frac{1}{2} < 0$, i.e. when $\alpha < -\frac{1}{2}$. In our endeavour to cover a wider range for the parameter α, we discuss in the present note the degree of approximation by partial sums of the series (1.1) under Lipschitz condition imposed on the generating function.

The n-th partial sum of the series (1.1) at the point $x = 0$ is given by

$$S_n(0) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0)$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \exp(-y)\,y^\alpha f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y)\,dy$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \exp(-y)\,y^\alpha f(y)\,L_n^{(\alpha+1)}(y)\,dy.$$

Using the orthogonality property of Laguerre functions, we easily have

$$(1.2) \qquad S_n(0) - f(0) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \exp(-y)\,y^\alpha[f(y) - f(0)]L_n^{(\alpha+1)}(y)\,dy$$

$$= \int_0^\infty \exp(-y)\,y^\alpha F(y)\,L_n^{(\alpha+1)}(y)\,dy,$$

where

$$F(y) = [f(y) - f(0)]\{\Gamma(\alpha+1)\}^{-1}.$$

Def. The function $f(y) \in \mathrm{Lip}\,(\gamma, p)$, $0 < \gamma \leq 1$, $p \geq 1$, if

$$\left(\int_0^t \left|\frac{F(y)}{y^\gamma}\right|^p dy\right)^{1/p} = O(1) \qquad 0 \leq t \leq a,$$

where a is a fixed number.

We prove the following

Theorem. *If* $f(x)\in \mathrm{Lip}\,(\gamma, p)$ *in every interval* $[0,\omega]$, ω *being a fixed number*, $0<\gamma\leqslant 1$, $p\geqslant 1$, *and if*

$$\int_{\omega}^{\infty} |F(y)|\, \exp(-y/2)\, dy < \infty \,, \tag{1.3}$$

then
$$|S_n(0)-f(0)| = O(n^{1/p-\gamma})\,,$$

provided α *is restricted by the conditions* $\alpha < 2/p - 1/2 - 2\gamma$, $-1/2<\alpha<1$, $\alpha > 1/p - 1 - \gamma$.

In the proof of the theorem we shall make use of the following order estimate of Laguerre functions (see Szegö [**3**]$_2$, pp. 174-176).

(i) If α is arbitrary and real; c, ω fixed positive constants, then as $n\to\infty$

$$L_n^{(\alpha)}(x) = x^{-\alpha/2-1/4}O(n^{\alpha/2-1/4}) \quad \text{for } \frac{c}{n}\leqslant x\leqslant \omega\,; \tag{1.4}$$

$$L_n^{(\alpha)}(x) = O(n^{\alpha}) \qquad \text{for } 0\leqslant x\leqslant \frac{c}{n}\,. \tag{1.5}$$

(ii) If α and λ are arbitrary and real, $a>0$, $0<\eta<4$, then for $n\to\infty$

$$\max \exp(-x/2)x^{\lambda}\,|L_n^{(\alpha)}(x)| \sim n^{Q}\,, \qquad \text{where} \tag{1.6}$$

$$Q = \begin{cases} \max\left(\lambda-\frac{1}{2}, \frac{\alpha}{2}-\frac{1}{4}\right) & a\leqslant x\leqslant (4-\eta)n \\ \max\left(\lambda-\frac{1}{3}, \frac{\alpha}{2}-\frac{1}{4}\right) & x\geqslant a\,, \end{cases} \tag{1.7}$$

the maxima being taken in the intervals indicated in the right hand members of (1.7).

2 - Proof of the theorem. From (1.2)

$$\begin{aligned} |S_n(0)-f(0) &\leqslant \int_0^{\infty} |F(y)|\, \exp(-y) y^{\alpha} |L_n^{(\alpha+1)}(y)|\, dy \\ &\leqslant \left(\int_0^{1/n} + \int_{1/n}^{\omega} + \int_{\omega}^{\infty}\right) |F(y)|\, \exp(-y) y^{\alpha} |L_n^{(\alpha+1)}(y)|\, dy \\ &\leqslant I_1 + I_2 + I_3\,, \qquad \text{say}\,. \end{aligned}$$

In the range $[0, 1/n]$, we use (1.5), i.e. $L_n^{(\alpha+1)}(y) = O(n^{\alpha+1})$, and employing Hölder's inequality, we obtain

$$I_1 \leqq \int_0^{1/n} |F(y)| \exp(-y) y^\alpha O(n^{\alpha+1})\, dy$$

$$= O(n^{\alpha+1}) \left(\int_0^{1/n} \left|\frac{F(y)}{y^\gamma}\right|^p dy\right)^{1/p} \left(\int_0^{1/n} |y^{\alpha+\gamma}|^q dy\right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1;$$

$$= O(n^{\alpha+1}) \frac{1}{n^{\alpha+\gamma+1/q}}, \quad \text{since } \alpha > \frac{1}{p} - 1 - \gamma = -\frac{1}{q} - \gamma,$$

$$= O(n^{1/p-\gamma}).$$

For the range $(1/n, \omega)$, we use the order estimate (1.4) and obtain

$$|L_n^{(\alpha+1)}(y)| = y^{-(\alpha+1)/2-1/4} \cdot O(n^{(\alpha+1)/2-1/4}) = O(y^{-\alpha/2-3/4}) n^{\alpha/2+1/4}.$$

Thus we have

$$I_2 = O(n^{\alpha/2+1/4}) \int_{1/n}^{\omega} |F(y)| \exp(-y) y^\alpha \cdot y^{-\alpha/2-3/4}\, dy$$

$$= O(n^{\alpha/2+1/4}) \left(\int_{1/n}^{\omega} \left|\frac{F(y)}{y^\gamma}\right|^p dy\right)^{1/p} \left(\int_{1/n}^{\omega} |y^{\alpha/2-3/4+\gamma}|^q dy\right)^{1/q}$$

$$= O(n^{\alpha/2+1/4}) \cdot O(1) \cdot O(n^{\gamma+\alpha/2-3/4+1/q})^{-1} \left(\text{since } \alpha < \frac{3}{2} - 2\gamma - \frac{2}{q} = \frac{2}{p} - \frac{1}{2} - 2\gamma\right),$$

$$= O(n^{1p-\gamma}).$$

Coming now to I_3, we have

$$I_3 = \int_{\omega}^{\infty} |F(y)| \exp(-y) y^\alpha |L_n^{(\alpha+1)}(y)|\, dy$$

$$= \int_{\omega}^{\infty} |F(y)| \exp(-y/2) [\exp(-y/2) y^\alpha |L_n^{(\alpha+1)}(y)|]\, dy.$$

From (1.7), it is clear that

$$|\exp(-y/2) y^\alpha L_n^{(\alpha+1)}(y)| \leqq C n^{\alpha/2+1/4},$$

because $\frac{\alpha+1}{2}-\frac{1}{4}>\alpha-\frac{1}{3}$ if $\alpha<\frac{7}{6}$. Consequently

$$I_3 = O[n^{\alpha/2+1/4}\cdot\int_{\omega}^{\infty}|F(y)|\,\exp(-y/2)\,dy] = O(n^{\alpha/2+1/4}),$$

from (1.3) $\quad I_3 = O(n^{1/p-\gamma}) \quad$ since $0<\frac{\alpha}{2}+\frac{1}{4}<(\frac{1}{p}-\gamma)$.

This completes the proof of the theorem.

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