Riv. Mat. Univ. Parma (4) 11 (1985), 25-29

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Invariant submanifolds of a quasi-K-Sasakian manifold (**)

Introduction

An invariant submanifold of a Sasakian manifold is Sasakian and minimal ([9], [10]). It is also known that an invariant submanifold of a K-contact manifold is K-contract and minimal ([2], $[5]_1$). The purpose of this paper is to show that similar results hold true for a more general class of manifolds, namely the class of quasi-K-Sasakian manifolds. We also obtain necessary and sufficient conditions in order that a manifold of this class be totally geodesic.

1 - Preliminaries

Let M be a manifold with an almost contact structure (F, ξ, η) and consider the manifold $M \times R$ (for the definitions and properties of almost contact structures we refer the reader to [1], [11]). We denote a vector field on $M \times R$ by $(X, a(\mathrm{d}/\mathrm{d}t))$, where X is tangent to M, t the coordinate of R and a is a C^∞ function on $M \times R$. S. Sasaki and Y. Hatakeyama [8] define an almost complex structure J on $M \times R$ by

$$J\,(X, a\,\frac{\mathrm{d}}{\mathrm{d}t}) = (FX - a\xi, \eta(X)\,\frac{\mathrm{d}}{\mathrm{d}t}) \; . \tag{1.1}$$

An almost contact structure is said to be *normal* if J is integrable.

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Now, if g is a Riemannian metric on the manifold M with a (F, ξ, η)-structure, we define a Riemannian metric on $M \times R$ by

$$h\left((X, a \frac{\mathrm{d}}{\mathrm{d}t}), (Y, b \frac{\mathrm{d}}{\mathrm{d}t})\right) = g(X, Y) + ab , \tag{1.2}$$

and another by

$$h^0 = \exp[2\sigma]h , \tag{1.3}$$

where $\sigma: M \times R \to R$ is defined by $\sigma(x, t) = t$ for all $(x, t) \in M \times R$.

In [**6**] J. Oubiña proved that a (F, ξ, η, g)-structure is a contact metric structure if and only if the structure (J, h^0) in $M \times R$ is almost Kaehlerian; it is a Sasakian structure if and only if (J, h^0) is a Kaehlerian structure.

An (F, ξ, η, g)-structure is called a *quasi-K-Sasakian structure* if (J, h^0) is a quasi-Kaehlerian structure. Thus, a (F, ξ, η, g)-structure is quasi-K-Sasakian if and only if

$$(\nabla_X F) Y + (\nabla_{FX} F) FY = 2g(X, Y)\xi + \eta(Y)\nabla_{FX}\xi - 2\eta(Y)X , \tag{1.4}$$

where $X, Y \in \chi(M)$ and ∇ is the covariant differentiation on M [**6**]. It follows that if (F, ξ, η, g) is a contact metric, K-contact metric or a Sasakian structure then (F, ξ, η, g) is a quasi-K-Sasakian structure. Moreover, in a quasi-K-Sasakian structure (F, ξ, η, g) we have

$$FX = \tfrac{1}{2}\left(F(\nabla_{FX}\xi) - \nabla_X\xi\right) . \tag{1.5}$$

2 - Invariant submanifolds of a quasi-K-Sasakian manifold

A submanifold M of an almost contact metric manifold $\tilde{M}$ with structure $(\tilde{F}, \xi, \tilde{\eta}, \tilde{g})$ is said to be *invariant* if $\tilde{F}X$ is tangent to M for any tangent vector X to M.

If the vector field ξ is never tangent to M, then the invariant submanifold M is an almost Hermitian manifold with the induced almost Hermitian structure (F, g), if ξ is always tangent to M, then M is an almost contact metric manifold with the induced almost contact metric structure (F, ξ, η, g), where $FX = \tilde{F}X$, $X \in \chi(M)$, ξ, η and g are the restrictions of $\xi, \tilde{\eta}$ and $\tilde{g}$ in M (see [**10**]).

Let M be an invariant submanifold of a quasi-K-Sasakian manifold $\tilde{M}$, then the vector field ξ is always tangent to M. In effect, if we suppose that M is an invariant submanifold of the quasi-K-Sasakian manifold $\tilde{M}$, with the vector field ξ never tangent to M, using the formula of Gauss (see [4], vol. II, p. 15) and (1.4), we obtain

$$(\nabla_X F)Y + (\nabla_{FX} F)FY = 0\,, \tag{2.1}$$

$$\alpha(X, FY) - \alpha(FX, Y) - \tilde{F}(\alpha(X, Y) + \alpha(FX, FY)) = g(X, Y)\xi\,, \tag{2.2}$$

for any vector fields X, $Y \in \chi(M)$, where α denote the second fundamental form of M. In particular, setting $X = Y$ in (2.2) we have

$$-\tilde{F}(\alpha(X, X) + \alpha(FX, FX)) = g(X, X)\xi\,,$$

which is a contradiction.

Theorem 1. *Any invariant submanifold M with induced structure (F, ξ, η, g) of a quasi-K-Sasakian manifold $\tilde{M}$ is also quasi-K-Sasakian.*

Proof. If $\tilde{M}$ is a quasi-K-Sasakian, then, by (1.4),

$$(\tilde{\nabla}_X \tilde{F})Y + (\tilde{\nabla}_{FX} \tilde{F})FY = 2\tilde{g}(X, Y)\xi + \tilde{\eta}(Y)\tilde{\nabla}_{FX}\xi - 2\tilde{\eta}(Y)X,$$

for any X, $Y \in \chi(M)$. Thus, using the formula of Gauss we obtain

$$(\nabla_X F)Y + (\nabla_{FX} F)FY = 2g(X, Y)\xi + \eta(Y)\nabla_{FX}\xi - 2\eta(Y)X,$$

for the tangential components, and

$$\alpha(X, FY) - \alpha(FX, Y) - \tilde{F}(\alpha(FX, FY) + \alpha(X, Y)) = 0\,,$$

for the normal components.

From the first identity we conclude that M is a quasi-K-Sasakian manifold.

Theorem 2. *Any invariant submanifold M of a quasi-K-Sasakian manifold is minimal and*

$$\alpha(FX, FY) = -\alpha(X, Y) \tag{2.3}$$

for any X, $Y \in \chi(M)$.

Proof. Since $\tilde{M}$ is a quasi-K-Sasakian manifold, we have

$$\alpha(X, FY) - \alpha(FX, Y) - \tilde{F}(\alpha(FX, FY) + \alpha(X, Y)) = 0\,, \quad \text{where } X,\ Y \in \chi(M)\,.$$

By symmetry, we obtain

$$\tilde{F}(\alpha(FX, FY) + \alpha(X, Y)) = 0.$$

Thus $\alpha(FX, FY) = -\alpha(X, Y)$, which proves our assertion.

Corollary 1. *Any invariant submanifold M of a contact metric manifold $\tilde{M}$ is minimal.*

Some results of H. Endo, S. Tanno, K. Yano-S. Ishihara follow easily from Theorem 2, (see [**2**], Th. 2.2, p. 155; [**9**], Prop. 4.1, p. 457; [**10**], Prop. 4.3, p. 361).

Theorem 3. *Let M be an invariant submanifold of a quasi-K-Sasakian manifold $\tilde{M}$. Then M is totally geodesic if and only if*

$$(\tilde{\nabla}_{FX}\alpha)(\xi, Y) = -(\tilde{\nabla}_X\alpha)(\xi, FY)$$

for any vector fields X and Y on M.

Proof. By (1.5), (2.3) and using the definition of the covariant derivative for the second fundamental form α of M, (see [4] p. 25), we have

$$2\alpha(FX, FY) = \alpha(\tilde{F}(\tilde{\nabla}_{FX}\xi) - \tilde{\nabla}_X\xi, FY) = -\alpha(\nabla_{FX}\xi, Y) - \alpha(\nabla_X\xi, FY)$$

$$= (\tilde{\nabla}_{FX}\alpha)(\xi, Y) + (\tilde{\nabla}_X\alpha)(\xi, FY). \quad \text{Thus}$$

$$\alpha(X, Y) = -\tfrac{1}{2}((\tilde{\nabla}_{FX}\alpha)(\xi, Y) + (\tilde{\nabla}_X\alpha)(\xi, FY)),$$

which proves our assertion.

A similar result of M. Kon [**5**]$_2$ follows easily from Theorem 3.

Theorem 4. *Let M be an invariant submanifold of a quasi-K-Sasakian manifold $\tilde{M}$ with constant sectional curvature. Then M is totally geodesis if and only if*

$$(\tilde{\nabla}_{FX}\alpha)(\xi, X) = 0 \qquad \textit{for any} \qquad X \in \chi(M).$$

Proof. By Theorem 3, we have

$$\alpha(X, X) = -\tfrac{1}{2}((\tilde{\nabla}_{FX}\alpha)(\xi, X) + (\tilde{\nabla}_X\alpha)(\xi, FX)). \tag{2.4}$$

But, if $\tilde{M}$ has constant sectional curvature then (see [4])

$$(2.5) \qquad (\tilde{\nabla}_X\alpha)(Y, Z) = (\tilde{\nabla}_Y\alpha)(X, Z) \qquad \text{for any } X, Y \in \chi(M) .$$

From (2.4) and (2.5), we conclude that

$$\alpha(X, X) = -(\tilde{\nabla}_{FX}\alpha)(\xi, X)$$

which proves our assertion.

Acknowledgment. I wish to express my hearty thanks to L. A. Cordero and J.A. Oubiña for several coments useful in the preparation of this paper. I am also very grateful to G. B. Rizza for his valuable suggestions.

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