Riv. Mat. Univ. Parma (4) **10*** (1984), 109-121

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A product formula for the volumes of tubes about Kähler submanifolds of complex euclidean space

1 - Introduction

The purpose of this note is to establish two formulas for the volumes of tubes about complex submanifolds of $\boldsymbol{C}^n$

$$V_P^{\boldsymbol{C}^n}(r) = \frac{1}{n!}\int_P \gamma(R^P)\wedge(\pi r^2 + F)^n\,, \tag{A}$$

$$V_{P\times Q}^{\boldsymbol{C}^{n+m}}\left(\sqrt{r_1^2+r_2^2}\right) = \sum_{b=0}^{m+n} V_P^{\boldsymbol{C}^b}(r_1)\; V_Q^{\boldsymbol{C}^{n+m-b}}(r_2)\,. \tag{B}$$

Here P and Q are Kähler manifolds for which the relevant integrals converge. In formula (A), $\gamma(R^P)$ denotes the total Chern form associated with the curvature tensor R^P of P, and F denotes the Kähler form of P. Because the differential form $\gamma(R^P)\wedge(\pi r^2+F)^n$ is not homogeneous, all terms of degree different from the dimension of P must be discarded when the integral on the right hand side of (A) is computed.

I shall show that when $P\subset \boldsymbol{C}^n$ as a Kähler submanifold, the volume $V_P^{\boldsymbol{C}^n}(r)$ of a tube of radius r about P in $\boldsymbol{C}^n$ is given by (A). However, it is important to observe that the right hand side of (A) makes sense for any Kähler manifold P for which the integral converges. When P is not given as a complex submanifold of $\boldsymbol{C}^n$, formula (*A*) may be regarded as a definition instead of a theorem, because both F and $\gamma(R^P)$ are intrinsic to P. With this interpretation both sides of (B) always make sense.

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For example when P is compact, $V_P^{C^0}(r)$ is just the Euler characteristic $\chi(P)$ of P. Thus (B) is a generalization of the well-known formula $\chi(P \times Q) = \chi(P)\chi(Q)$.

There are two generalizations of formula (A). The first treats Kähler submanifolds of a Kähler manifold M whose sectional curvature K^M has constant sign.

Theorem 1.1. *Let P be a Kähler submanifold of a complete Kähler manifold M whose sectional curvature K^M is always nonnegative or always nonpositive. Assume that P is relatively compact and topologically embedded, and that $r > 0$ is not larger than the distance from P to its nearest focal point, Then*

(i) $K^M \geqslant 0$ *implies*

$$V_P^M(r) \leqslant \frac{1}{n!}\int_P \gamma(R^P - R^M)\wedge(\pi r^2 + F)^n \leqslant \frac{(\pi r^2)^{n-q}}{(n-q)!}\operatorname{vol}(P)\ ,$$

(ii) $K^M \leqslant 0$ *implies*

$$V_P^M(r) \geqslant \frac{1}{n!}\int_P \gamma(R^P - R^M)\wedge(\pi r^2 + F)^n\ .$$

Note that if R is any tensor field having the same symmetries as the curvature tensor of a Kähler manifold it is possible to define the total Chern form $\gamma(R)$. In particular $\gamma(R^P - R^M)$ makes sense, where R^P and R^M are the curvature operators of P and M.

The second generalization of formula (A) treats an arbitrary oriented 2-dimensional submanifold P of $\boldsymbol{R}^n$. Let χ be the Euler form of P. Thus $\chi = (2\pi)^{-1}K^P\omega$, where K^P is the sectional curvature of P and ω is the Riemannian volume form determined by the orientation of P. Of course when P is compact, the Gauss Bonnet theorem states that $\int_P \chi = \chi(P)$. The following formula holds

$$\text{(C)}\qquad V_P^{R^n}(r) = \frac{(\pi r^2)^{n/2-1}}{(n/2-1)!}\int_P \left\{\omega + \frac{2\pi r^2 \chi}{n}\right\} = \frac{(\pi r^2)^{n/2}}{(n/2)!}\int_P (1+\chi)\wedge\left(1+\frac{\omega}{\pi r^2}\right)^{n/2}.$$

If $P \subset \boldsymbol{R}^n$ the $V_P^{R^n}(r)$ is the volume of a tube of radius r about P; otherwise (C) may be taken as a definition. More generally

Theorem 1.2. *Let M be a complete Riemannian manifold whose sectional curvature has constant sign. Assume that P is relatively compact and topologically* *embedded, and that $r > 0$ is not larger than the distance between P and its nearest focal point. Then $K^M \geqslant 0$ implies*

$$(1.1) \qquad V_P^M(r) \leqslant \frac{(\pi r^2)^{n/2-1}}{(n/2-1)!} \int_P \left\{\omega + \frac{r^2}{n}(2\pi\chi - K^M\omega)\right\}.$$

If $K^M \leqslant 0$ the inequality is reversed in (1.1).

Formulas (A) and (C) may be regarded as sharpened versions of the Weyl tube formula [4] for the volumes of tubes in $\boldsymbol{R}^n$

$$(1.2) \qquad V_P^{\boldsymbol{R}^n}(r) = \frac{(\pi r^2)^{(n-q)/2}}{((n-q)/2)!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P)\, r^{2c}}{(n-q+2)\ \dots\ (n-q+2c)}, \quad q = \dim P.$$

Weyl showed that the $k_{2c}(R^P)$ are integrals of certain functions of the curvature operator R^P of P. Hence $V_P^{\boldsymbol{R}^n}(r)$ does not depend on the particular way P is immersed in $\boldsymbol{R}^n$ but only on the Riemannian metric of P. For this reason if P is not given as a submanifold of $\boldsymbol{R}^n$, (1.2) can be used as a definition provided all of the integrals converge. Thus for each n one can associate with the Riemannian manifold P the function $r \to V_P^{\boldsymbol{R}^n}(r)$.

Now $r \to V_P^{\boldsymbol{R}^n}(r)$ is a metric invariant, but can one say more? The answer is yes for Kähler manifolds. Recall (see for example [**1**], p. 106) that a Kähler deformation consists of a change of Kähler form F to $F + id'd''f$, where f is any differentiable real valued function on M.

Theorem 1.3. *Let P be a compact Kähler manifold with Kähler form F, and define $V_P^{\boldsymbol{C}^n}(r)$ by formula* (A). *Then $V_P^{\boldsymbol{C}^n}(r)$ is the same for all Kähler deformations of F.*

For compact oriented two dimensional manifolds the situation is even better.

Theorem 1.4. *Let P and Q be compact two dimensional manifolds. Fix n. Then the following conditions are equivalent:*

(i) *P and Q are homeomorphic and* $\operatorname{vol}(P) = \operatorname{vol}(Q)$;

(ii) $V_P^{\boldsymbol{R}^n}(r) \equiv V_Q^{\boldsymbol{R}^n}(r)$.

2 - Fermi coordinates and Fermi fields

The computations necessary for formula (A) and Theorem 1.1 are most conveniently performed in terms of special vector fields called Fermi fields [**5**]. To explain this notion let P be a submanifold of a Riemannian manifold M

and let $\exp_\nu$ be the exponential map of the normal bundle ν. For $p \in P$ choose a coordinate system $(y_1, \ldots, y_q)$ in a neighborhood of p in P and choose orthonormal sections $E_{q+1}, \ldots, E_n$ of ν. Then the *Fermi coordinates* of $P \subset M$ (relative to $(y_1, \ldots, y_q)$ and $E_{q+1}, \ldots, E_n$) are given by

$$x_a(\exp_\nu (\sum t_j E_j(m)) = y_a(m) \qquad a = 1, \ldots, q\,,$$
$$x_i(\exp_\nu (\sum t_j E_j(m)) = t_i \qquad i = q+1, \ldots, n\,.$$

Definition. Let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates for $P \subset M$. A vector field on M is said to be a *Fermi field* if it is a constant linear combination of the coordinate vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_n$. There are two kinds of Fermi fields: *tangential* (those that are constant linear combinations of $\partial/\partial x_1, \ldots, \partial/\partial x_q$) and *normal* (those that are constant linear combinations of $\partial/\partial x_{q+1}, \ldots, \partial/\partial x_n$). The spaces of tangential, normal, and all Fermi fields will be denoted by $\mathscr{X}(P, p)^T$, $\mathscr{X}(P, p)^\perp$, and $\mathscr{X}(P, p)$ respectively.

In addition to Fermi fields it will be useful to consider the function σ and the vector field N defined in a neighborhood of P in M as follows

$$\sigma(m) = d(m, P)\,, \qquad N_{\gamma(s)} = \gamma'(s)\,,$$

where d is the distance function of M and γ is any unit speed geodesic normal to P. In terms of Fermi coordinates and Fermi fields

$$\sigma^2 = \sum_{i=q+1}^{n} x_i^2 \quad \text{and} \quad N = \sum_{i=q+1}^{n} \frac{x_i}{\sigma} \frac{\partial}{\partial x_i}\,.$$

The calculus of σ, N and the Fermi fields is developped in [**5**]. The key facts are summarized in the following lemma.

Lemma 2.1. *Let* $X, Y \in \mathscr{X}(P, p)^\perp$, $A, B \in \mathscr{X}(P, p)^T$. *Then*

(i) $\nabla_N N = 0$

(ii) $N = \operatorname{grad} \sigma$ (*generalized Gauss Lemma*)

(iii) $[X, Y] = [A, B] = [X, A] = [N, A] = 0$

(iv) $[N, \sigma X] = (X\sigma)N$

(v) $\nabla_N \nabla_N U + R^M_{NU} N = 0$ *for* $U = A + \sigma X$ (*Jacobi's equation*).

Proof. (i) is obvious from the definition and so is (iii). For the proof of (ii), which is somewhat complicated, see [**5**]. Note that (ii) implies that $N(\sigma) = 1$. For (iv) one calculates as follows

$$[N, \sigma X] = (N\sigma)X + \sigma \sum_{i=q+1}^{n} X\left(\frac{x_i}{\sigma}\right) \frac{\partial}{\partial x_i} = X + \sigma \sum_{i=q+1}^{n} \left\{-\frac{1}{\sigma} X(x_i) + \frac{x_i}{\sigma^2} X(\sigma)\right\} \frac{\partial}{\partial x_i}$$

$$= X - \sum X(x_i) \frac{\partial}{\partial x_i} + X(\sigma) \sum \frac{x_i}{\sigma} \frac{\partial}{\partial x_i} = X(\sigma) N .$$

Finally for (v) the computations are

$$\nabla_N \nabla_N(\sigma X) = \nabla_N[N, \sigma X] + \nabla_N \nabla_{\sigma X} N = \nabla_N\big((X\sigma)N\big) - R^M_{N\sigma X} N + \nabla_{[N,\sigma X]} N$$

$$= (NX\sigma)N - R^M_{N\sigma X} N = - R^M_{N\sigma X} N$$

and $$\nabla_N \nabla_N A = \nabla_N \nabla_A N = - R^M_{NA} N .$$

3 - The Riccati equation for the second fundamental forms

Let S be the (1,1) tensor field defined on a neighborhood of P in M by $SU = - \nabla_U N$.

Then geometrically the restriction of S to a tubular hypersurface P_t at a distance t from P is just the second fundamental form of P. Also let R_N be the (1,1) tensor field given by $R_N U = R^M_{NU} N$.

Lemma 3.1. $\nabla_N(S) = S^2 + R_N$.

Proof. Let $m \in M$ be near but not on P. Then the tangent space M_m is spanned by vectors of the form $(A + \sigma X)_m$ where A is a tangential Fermi field and X is a normal Fermi field. Thus it suffices to prove

$$\nabla_N(S)\, U = (S^2 + R_N)\, U ,$$

where $U = A + \sigma X$. In fact

$$\nabla_N(S)\, U = \nabla_N(SU) - S\nabla_N U = - \nabla_N \nabla_N U - S[N, U] + S^2 U$$

$$= - [\nabla_N, \nabla_U] N + \nabla_{[N,U]} N + S^2 U = - R_N U + S^2 U .$$

Denote by $S(t)$ the second fundamental form of the tubular hypersurface P_t. Also let $R(t)$ denote the restriction of R_N to P_t. Then Lemma 3.1 can be reinterpreted as

Corollary 3.2. $S'(t) = S(t)^2 + R(t)$.

Next let ω be a volume form on M defined near P with $\|\omega\| = 1$, and let and let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates such that $\omega(\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_n) > 0$. For $u \in P_p^\perp$ with $\|u\| = 1$ put

$$\theta_u(t) = \omega\,(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n})\,(\gamma(t))\,,$$

where $t \to \gamma(t)$ is the unit speed geodesic in M with $\gamma(0) = p$ and $\gamma'(0) = u$.

Lemma 3.3. $(\mathrm{d}/\mathrm{d}t) \ln \theta_u(t) = -\big((n-q-1)/t + \operatorname{tr} S(t)\big)$.

Proof. It is clear that θ_u does not depend on the choice of (oriented) Fermi coordinates at $p = \gamma(0)$. Thus the system may be chosen so that

$$\left.\frac{\partial}{\partial x_{q+1}}\right|_p = \gamma'(0)\,.$$

Then $\{\partial/\partial x_1|_p, \ldots, \partial/\partial x_n|_p\}$ forms an orthonormal basis for M_p. Write

$$A_a = \left.\frac{\partial}{\partial x_a}\right|_\gamma, \quad N = \left.\frac{\partial}{\partial x_{q+1}}\right|_\gamma, \quad X_i = \left.\frac{\partial}{\partial x_i}\right|_\gamma,$$

for $a = 1, \ldots, q$ and $i = q+2, \ldots, n$. Then

$$\theta_u(t) = \omega(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n)(t)\,,$$

$$\begin{aligned}
\theta_u'(t) &= N\omega(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+1} \wedge \ldots \wedge X_n)(t) \\
&= \{\sum_{a=1}^{q} \omega(A_1 \wedge \ldots \wedge \nabla_N A_a \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n) \\
&\qquad + \sum_{i=q+2}^{n} \omega(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge \nabla_N X_i \wedge \ldots \wedge X_n)\}(t) \\
&= \{\sum_{a=1}^{q} \omega(A_1 \wedge \ldots \wedge \nabla_{A_a} N \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n) \\
&\qquad + \sum_{i=q+2}^{n} \omega\big(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge (\nabla_{X_i} N + [N, X_i]) \wedge \ldots \wedge X_n\big)\}(t) \\
&= \{\sum_{a=1}^{q} \omega(A_1 \wedge \ldots \wedge (-SA_a) \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n) \\
&\qquad + \sum_{i=q+2}^{n} \omega\,(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge (-SX_i - \frac{1}{t} X_i) \wedge \ldots \wedge X_n)\}\,(t) \\
&= -\big(\frac{n-q-1}{t} + \operatorname{tr} S(t)\big)\,\theta_u(t)\,.
\end{aligned}$$

Hence the lemma follows.

For certain simple differentiable manifolds M (and arbitrary P in M) it is possible to solve explicitly the differential system $S'(t) = S(t)^2 + R(t)$. Consequently $\theta_u(t)$ can also be found explicitly. For example when M is flat one has

Lemma 3.4. *Let P be a submanifold of a flat space M. Then along the geodesic γ normal to P with $\gamma(0) = p$ and $\gamma'(0) = u$ (where $u \in P_p^\perp$ and $\|u\| = 1$)*

$$S(t) = \begin{pmatrix} T_u(I - tT_u)^{-1} & 0 \\ 0 & -t^{-1}I \end{pmatrix}. \tag{3.1}$$

Proof. Choose an orthonormal frame field $\{E_1(t), \ldots, E_n(t)\}$ along γ that diagonalizes $S(t)$ at each t

$$S(t)E_\alpha(t) = \varkappa_\alpha(t)E_\alpha(t), \qquad \alpha \neq q+1$$

Here it may be assumed that $\{E_1(0), \ldots, E_q(0)\}$ and $\{E_{q+1}(0), \ldots, E_n(0)\}$ are orthonormal bases of P_p and $P_p^\perp$ respectively, and that $E_{q+1}(0) = u$. It follows from Corollary 3.2 and the assumption that M is flat that

$$\varkappa_\alpha'(t) = \varkappa_\alpha(t)^2 \qquad \alpha = 1, \ldots, q, q+2, \ldots, n, \tag{3.2}$$

except possibly for a finite number of points where $\varkappa_\alpha$ coincides with some $\varkappa_\beta$. It is clear that $|\varkappa_{q+2}(t)|, \ldots, |\varkappa_n(t)|$ become infinite as $t \to 0$. Moreover, each $t \to \varkappa_\alpha(t)$ is an increasing function because of (3.2). It follows that $\varkappa_{q+2}(0) = \ldots = \varkappa_n(0) = -\infty$. Furthermore $\varkappa_1(0), \ldots, \varkappa_q(0)$ are finite. Thus solving (3.2) and using these initial conditions one finds that

$$\begin{aligned} \varkappa_a(t) &= \varkappa_a(0)(1 - t\varkappa_a(0))^{-1} \qquad & a = 1, \ldots, q, \\ \varkappa_i(t) &= -t^{-1} & i = q+2, \ldots, n. \end{aligned} \tag{3.3}$$

At least (3.3) holds where there is no differentiability problem. Moreover the submanifold P can always be deformed slightly so that all of the $\varkappa_a(t)$'s are distinct. Then (3.3) holds for the deformed submanifold without exception. Since (3.3) is a relation not involving derivatives, it must also hold for the original submanifold. Finally (3.3) can obviously be rewritten as (3.1).

Lemma 3.5. *Let P be a submanifold of a flat space M. Then*

$$\theta_u(t) = \det(I - tT_u). \tag{3.4}$$

Proof. Using lemmas (3.3) and (3.4) one finds

$$(3.5)\qquad \frac{\theta_u'(t)}{\theta_u(t)} = -\Big(\frac{n-q-1}{t} + \operatorname{tr} S(t)\Big) = \operatorname{tr}\big(T_u(I-tT_u)^{-1}\big) = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\ln(I-tT_u)\,.$$

Using the relation $e^{\operatorname{tr} A} = \det(e^A)$, (3.5) can be written as

$$(3.6)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\ln\,\theta_u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\det(I-tT_u)\,.$$

Since $\theta_u(0)=1$ (3.6) may be integrated to yield (3.4).

In **[5]** generalizations of lemmas 3.4 and 3.5 to nonflat Riemannian manifolds are given. Let M be a complete Riemannian manifold and denote by K^M the sectional curvature of M. Assume that K^M is always nonnegative or nonpositive. Then the generalization of equation (3.2) is

$$(3.7)\qquad \varkappa_\alpha'(t) \leqslant \varkappa_\alpha(t)^2 \quad \text{for } K^M \geqslant 0\,, \qquad \varkappa_\alpha'(t) \geqslant \varkappa_\alpha(t)^2 \quad \text{for } K^M \leqslant 0\,.$$

This is a consequence of Corollary 3.2. Although the equation $S'(t)=S(t)^2+R(t)$ in general cannot be explicitly solved, at least inequalities for the principal curvatures can be obtained. A generalization of (3.3) (which follows from (3.7)) is

$$(3.8)\qquad \begin{aligned} &\varkappa_a(t) \leqslant \varkappa_a(0)\,(1-t\varkappa_a(0))^{-1} && a = 1, \dots, q\,,\\ &\varkappa_i(t) \leqslant (-t)^{-1} && i = q+2, \dots, n\,. \end{aligned}$$

for $K^M \geqslant 0$. When $K^M \leqslant 0$ the inequalities are reversed. The from (3.8) follows a generalization of Lemma 3.5.

Lemma 3.6. *Let P be a submanifold of a complete Riemannian manifold M. Assume K^M is always nonnegative or always nonpositive. Then*

$$\theta_u(t) \leqslant \det(I-tT_u) \quad \text{for } K^M \geqslant 0\,, \qquad \theta_u(t) \geqslant \det(I-tT_u) \quad \text{for } K^M \leqslant 0\,.$$

4 - Relations between Chern forms and the change of volume factor

Let P be an almost Hermitian manifold of complex dimension q, and let R be any tensor field on P that has the same symmetries as the curvature tensor of a Kähler manifold. Let $\{E_1, JE_1, \dots, E_q, JE_q\}$ be a local frame field on P. Then the complex curvature forms of R with respect to this frame field are the 2-forms defined by

$$\Xi_{ab}(X \wedge Y) = R_{E_a E_b XY} - \sqrt{-1}\, R_{E_a JE_b XY}$$

for complex vector fields X, Y on M. Then by definition the total Chern form $\gamma(R)$ of R is

$$\gamma(R) = \det\left(\delta_{ab} + \frac{\sqrt{-1}}{2\pi}\, \Xi_{ab}\right) \tag{4.1}$$

(see for example [4] for this notation).

The individual Chern forms are determined by means of the decomposition

$$\gamma(R) = 1 + \gamma_1(R) + \dots + \gamma_q(R)\,, \qquad \gamma_c(R) \in \wedge^{2c}(P)\,.$$

It is also possible to form the c^{th} power of R (see for example [2], [3]). The definition can be given inductively via the formulas $R^0 = 1$ and

$$R^c(X_1 \wedge \dots \wedge X_{2c})(Y_1 \wedge \dots \wedge Y_{2c}) = \sum_{i,j,k,l=1}^{2c} (-1)^{i+j+k+l} R_{X_i X_j Y_k Y_l}$$
$$R^{c-1}(X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_{2c})(Y_1 \wedge \dots \wedge \hat{Y}_k \wedge \dots \wedge \hat{Y}_l \wedge \dots \wedge Y_{2c})\,.$$

Then the complete contraction of R^c is

$$C^{2c}(R^c) = \sum_{a_1 \dots a_c = 1}^{2q} R^c(E_{a_1} \wedge \dots \wedge E_{a_{2c}})(E_{a_1} \wedge \dots \wedge E_{a_{2c}})\,,$$

where $\{E_1, \dots, E_{2q}\}$ is any local orthonormal frame on P. In [6] it is shown that it is possible to express $C^{2c}(R^c)$ in terms of the Chern forms of R and the Kähler form F.

Lemma 4.1. *Let R be a tensor field on an almost Hermitian manifold M that has all of the symmetries of the curvature tensor field of a Kähler manifold. Then*

$$\frac{(q-c)!}{c!(2c)!}\, C^{2c}(R^c) = (2\pi)^c \left\langle \gamma_c(R) \wedge F^{q-c}, \frac{1}{q!} F^q \right\rangle. \tag{4.2}$$

Proof of formula (A). Let $S^{2n-2q-1}(1)$ denote the unit sphere in $P_p^{\perp}$. From (3.4) it follows that

$$\begin{aligned}(4.3)\qquad &\int_{S^{2n-2q-1}(1)} \theta_u(t)\,\mathrm{d}u \\ &= \int_{S^{2n-2q-1}(1)} \det(I - tT_u)\,\mathrm{d}u = 2\pi \sum_{c=0}^{q} \frac{C^{2c}\big((R^P)^c\big)\,t^{2c}}{c!(2c)!2^c\Gamma(n-q+c)}\,.\end{aligned}$$

The details are given in **[5]**. In fact this is the heart of the proof of the Weyl tube formula. In **[5]** it is also shown that

$$(4.4)\qquad V_P^M(r) = \int_0^r \int_P \int_{S^{2n-2q-1}(1)} \theta_u(t)\,\mathrm{d}u\,\mathrm{d}P\,\mathrm{d}t\,.$$

In (4.4) M is an arbitrary Riemannian manifold. Together (4.3) and (4.4) yield

$$(4.5)\qquad V_P^{C^n}(r) = (\pi r^2)^{n-q} \sum_{c=0}^{q} \frac{r^{2c}}{c!(2c)!\,2^c(n-q+c)!} \int_P C^{2c}\big((R^P)^c\big)\,\mathrm{d}P\,.$$

Here P is any submanifold of real dimension $2q$ for which $\int_P C^{2c}\big((R^P)^c\big)\,\mathrm{d}P$ converges for $c = 0, \ldots, q$. Formula (4.5) is just the Weyl tube formula for a submanifold of an even dimensional Euclidean space. When P is a complex submanifold of $\boldsymbol{C}^n$, (4.2) and (4.5) combine to yield

$$\begin{aligned}V_P^{C^n}(r) &= \sum_{c=0}^{q} \frac{(\pi r^2)^{n-q+c}}{(q-c)!(n-q+c)!} \int_P \langle \gamma_c(R^P)\wedge F^{q-c}, \frac{1}{q!}F^q\rangle\,\mathrm{d}P \\ (4.6)\qquad &= \sum_{c=0}^{q} \frac{(\pi r^2)^{n-q+c}}{(q-c)!(n-q+c)!} \int_P \gamma_c(R^P)\wedge F^{q-c}\,,\end{aligned}$$

because the volume element of P is $\mathrm{d}P = (1/q!)F^q$.

On the other hand when the expression $\gamma_c(R^P)\wedge(\pi r^2 + F)^n$ is integrated over P all terms not of degree $2q$ can be eliminated. Hence (using the binomial theorem) it follows that

$$\begin{aligned}\frac{1}{n!}\int_P \gamma(R^P)\wedge(\pi r^2 + F)^n &= \sum_{c=0}^{q} \sum_{k=0}^{n} \frac{(\pi r^2)^{n-k}}{k!(n-k)!} \int_P \gamma_c(R^P)\wedge F^k \\ (4.7)\qquad &= \sum_{c=0}^{q} \frac{(\pi r^2)^{n-q+c}}{(q-c)!(n-q+c)!} \int_P \gamma_c(R^P)\wedge F^{q-c}\,.\end{aligned}$$

Now formula (A) follows from (4.6) and (4.7).

Proof of Theorem 1.1. Assume $K^M \geqslant 0$ (the proof when $K^M \leqslant 0$ is analogous). Just as (3.4) implies (4.3) it follows from Lemma 3.6 that

$$\begin{aligned}\int_{S^{2n-2q-1}(1)} \theta_n(t)\,\mathrm{d}u &\leqslant \int_{S^{2n-2q-1}(1)} \det(I - tT_u)\,\mathrm{d}u \\ &= 2\pi^{n-q} \sum_{c=0}^{q} \frac{C^{2c}\big((R^P - R^M)^c\big)\, t^{2c}}{c!(2c)!\, 2^c \Gamma(n-q+c)}\,.\end{aligned} \tag{4.8}$$

Now the rest of the proof proceeds exactly as the proof of formula (A). The only differences are that $-R^M$ is used instead of R^P and $=$ is replaced by $\leqslant$.

Proof of formula (C). For $\dim P = 2$ the Weyl tube formula reduces to

$$V_P^{\mathbf{R}^n}(r) = \frac{(\pi r^2)^{n/2-1}}{(n/2-1)!}\left\{k_0(R^P) + \frac{k_2(R^P)\, r^2}{n}\right\}. \tag{4.9}$$

Here $k_0(R^P) = \operatorname{vol}(P)$ and

$$k_2(R^P) = \tfrac{1}{2}\int_P \tau(R^P)\,\mathrm{d}P = \int_P K^P\,\mathrm{d}P = 2\pi\chi(P) = 2\pi\int_P \chi$$

by the Gauss Bonnet theorem. Hence (4.9) can be rewritten as

$$\begin{aligned}V_P^{\mathbf{R}^n}(r) &= \frac{(\pi r^2)^{n/2-1}}{(n/2-1)!}\int_P \left\{\omega + \frac{2\pi\chi r^2}{n}\right\} \\ &= \frac{(\pi r^2)^{n/2}}{(n/2)!}\int_P (1+\chi)\wedge\Big(1 + \frac{\omega}{\pi r^2}\Big)^{n/2}.\end{aligned} \tag{4.10}$$

Proof of Theorem 1.2. The proof is similar but one use the fact that $k_2(R^P - R^M) = k_2(R^P) - k_2(R^M) = \int_P (2\pi\chi - K^M\omega)$.

5 - Proof of formula (*B*)

Let P and Q be complex submanifolds of $\boldsymbol{C}^n$ and $\boldsymbol{C}^m$ with complex dimensions p and q respectively. Let F_1 and F_2 be the Kähler forms and R^P and R^Q the curvature tensor fields. Then the product Kähler manifold $P\times Q$ has Kähler form $F = F_1 + F_2$ and curvature tensor field $R^{P\times Q} = R^P + R^Q$. Moreover the

Whitney sum theorem implies the following relation between the various Chern forms

$$(5.1) \qquad \gamma(R^{P\times Q}) = \gamma(R^P)\wedge\gamma(R^Q)\,.$$

The Whitney sum theorem is usually stated for Chern classes instead of Chern forms, so that strictly speaking (5.1) is a refinement. In any case (5.1) is easily established using (4.1) and the fact that

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det A \det B\,.$$

Using formulas (A) and (5.1) the proof of formula (B) is as follows

$$V_{P\times Q}^{C^{n+m}}\left(\sqrt{r_1^2 + r_2^2}\,\right) = \frac{1}{(m+n)!}\int_{P\times Q} \gamma(R^{P\times Q})\wedge\left(\pi(r_1^2 + r_2^2 + F_1 + F_2)\right)^{m+n}$$

$$= \int_{P\times Q} \gamma(R^P)\wedge\gamma(R^Q)\wedge \sum_{b=0}^{m+n} \frac{(\pi(r_1^2 + F_1))^b\wedge(\pi(r_2^2 + F_2))^{m+n-b}}{b!(m+n-b)!}$$

$$= \sum_{b=0}^{m+n}\left\{\frac{1}{b!}\int_P \gamma(R^P)\wedge(\pi(r_1^2 + F_1)^b)\right\}\left\{\frac{1}{(m+n-b)!}\int_Q \gamma(R^Q)\wedge(\pi(r_2^2 + F_2))^{m+n-b}\right\}$$

$$= \sum_{b=0}^{m+n} V_P^{C^b}(r_1)\; V_Q^{C^{m+n-b}}(r_2).$$

6 - Deformations

Proof of Theorem 1.3. The proof is elementary and essentially the same as that of [1], p. 117. Let $F \to F + i\,d'd''f$ be a Kähler deformation. The metric also changes, but at any rate the total Chern form of the new metric has the form $\gamma(R^P) + d\alpha$. But then

$$(\gamma(R^P) + d\alpha)\wedge(\pi r^2 + F + i\,d'd''f) = \gamma(R^P)\wedge(\pi r^2 + F)^n + d\varepsilon$$

for some (nonhomogeneous) differential form ε. Thus by Stokes' theorem

$$V_P^{C^n}(r) = \frac{1}{n!}\int_P (\gamma(R^P) + d\alpha)\wedge(\pi r^2 + F + i d'd''f)^n\,.$$

Proof of Theorem 1.4. If P and Q are homeomorphic then $\chi(P) = \chi(Q)$. If in addition $\mathrm{vol}(P) = \mathrm{vol}(Q)$ then formula (C) implies $V_P^{R^n}(r) \equiv V_Q^{C^n}(r)$.

Conversely if $V_P^{R^n}(r) \equiv V_Q^{R^n}(r)$, then $\mathrm{vol}(P) = \mathrm{vol}(Q)$ and $\chi(P) = \chi(Q)$. Because P and Q are both orientable it follows that P and Q are homeomorphic.

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