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ALBRECHT DOLD (*)

Combinatorial and geometric fixed point theory

Introduction

Combinatorial fixed point theory in the sense of this lecture is the fixed point theory of self-maps $\varphi: X \to X$ of finite sets. It is concerned with the number of fixed points, or periodic points, of φ, or of maps associated with φ. *Geometric fixed point theory* in the sense of this lecture deals with continuous maps $f: V \to Y$, where Y is a euclidean neighborhood retract (ENR), $V \subset Y$ is an open subset, and f has compact fixed point set $\text{Fix}(f) = \{v \in V \mid f(v) = v\}$. Two such maps are said to be *equivalent* ($\sim$) if there is a third such map $F: W \to Z$ which lies over the interval $[0, 1] \subset \boldsymbol{R}$ (i.e. $p: Z \to [0, 1]$ is an $\text{ENR}_{[0,1]}$ in the sense of [4], $pF = p \mid W$), and the parts of F over 0 resp. 1 are homeomorphic to f_0 resp. f_1. Geometric fixed point theory then is concerned with properties of f which are invariant with respect to $\sim$. One, and in some sense the only such invariant is the Hopf index $I(f) \in \boldsymbol{Z}$. It should be thought of as *counting* the fixed points of f in an $\sim$invariant way. Of course, $I(f) = |\text{Fix}(f)|$ = cardinality of $\text{Fix}(f)$ if Y is finite.

The purpose of this lecture is to show that many (not all) basic problems of geometric fixed point theory can be reduced to the combinatorial case. The combinatorial case is, of course, much easier to handle although it may still present tough problems. We begin in **1** with some easy but typical results on the Euler-characteristic (= index of the identity map). In **2** we discuss the indices of iterated maps (referring to [5] for more detail). The same subject is presented in **3** in a more formal and systematic way, where we introduce and discuss the appropriate Grothendieck rings (similar rings will be used again

(*) Address: Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, 6900 Heidelberg, Deutsche Bundesrepublik.

in **5** and **6**). In **4** we show how the indices $\{I(f^k)\}_{k=1,\dots,n}$ of the first n iterates of f are related to the indices $\{I(\mathrm{SP}^k f)\}_{k=1,\dots,n}$ of the symmetric power maps. In **5** we describe the possible invariants of G-equivariant maps in terms of the Burnside ring of G (for finite groups G). Parametrised fixed point theory and its Grothendieck ring is described in **6**; whether and how this can be reduced to a combinatorial theory is formulated as an open problem.

1 - Euler-characteristic of symmetric powers

The Euler-characteristic $e(Y)$ of a compact ENR Y is (by definition, more or less) the index $I(\mathrm{id}_Y)$ of the identity map. According to the introduction it *counts* the points of Y is an $\sim$invariant way (compare also [**12**]). The n-th symmetric power $\mathrm{SP}^n Y$ is the quotient of the cartesian power Y^n by the action of the symmetric group $S(n)$. More generally, for every subgroup $\pi \subset S(n)$, one defines the π-power $P^\pi Y = Y^n/\pi$. If X is a finite set, then the cardinality $|P^\pi X|$ can be calculated in terms of $|X|$,

$$|P^\pi X| = p_\pi(|X|)\,, \tag{1.1}$$

where $p_\pi(x) \in Q[x]$ is a polynomial of degree n with rational coefficients (an integral combination of binomial coefficients $\binom{x+k}{k}$, $0 \leqslant k \leqslant n$, since it assumes integral values on $\boldsymbol{Z}$). Our general principle of reducing to the combinatorial case leads to the formula

$$e(P^\pi Y) = p_\pi(e(Y))\,, \tag{1.2}$$

which is valid, indeed, *for all compact* ENR*s* Y.

A proof with algebraic K-theory can be found in [**3**]. It can also be proved directly using the Künneth-formula for Y^n and some linear algebra. Still another proof is indicated in **4** (example after (4.2)).

For $\pi = S(n)$ one finds

$$p_{S(n)}(x) = \binom{x+n-1}{n} = (-1)^n \binom{-x}{n} \tag{1.3}$$

hence

$$e(\mathrm{SP}^n Y) = (-1)^n \binom{-e(Y)}{n}\,.$$

It is convenient to regard the whole sequence $\{e(\mathrm{SP}^n Y)\}_{n=0,1,\dots}$ as a formal power series,

$$\sum_{n=0}^{\infty} e(\mathrm{SP}^n\, Y)\, t^n = (1-t)^{-e(Y)}, \tag{1.4}$$

for all compact ENRs Y. This is a special case of (4.2).

2 - Indices of iterated maps

If Y is an ENR, $V \subset Y$ an open subset and $f\colon V \to Y$ a continuous map, then we define the iterates $f^k\colon V_k \to Y$ inductively by $f^1 = f$, and $V_k = f^{-1}(V_{k-1})$, $f^k(v) = f^{k-1}(f(v))$ for $k > 1$. The indices $I(f_k)$ are defined if Fix (f^k) is compact. If Fix (f^n) is compact then so is Fix (f^k) for all k which divide n. We can then define an integer $I_n(f)$ by the following Möbius formula

$$I_n(f) = \sum_{\tau \subset P(n)} (-1)^{|\tau|} I(f^{n:\tau}), \tag{2.1}$$

where $P(n)$ is the set of all primes which divide n, the sum extends over all subsets τ of $P(n)$, $|\tau|$ = cardinality of τ, and $n\colon \tau = n(\prod_{p \in \tau} p)^{-1} = n$ divided by all $p \in \tau$.

2.2 Lemma. *If X is a finite set and $f\colon X \to X$ is a map, then $I_n(f)$ is the number of points of period exactly n, i.e. $I_n(f)$ is the cardinality of*

$$\{x \mid f^n(x) = x \text{ but } f^m(x) \neq x \text{ for } m < n\}.$$

This is an easy exercise (cf. [**1**], App.). It has the following

2.3 Corollary. $I_n(f) \equiv 0 \pmod n$. *In fact, $(1/n) I_n(f)$ is the number of f-orbits of length n.*

The lemma and its corollary make sense in geometric fixed point theory (where fixed points are counted by their index). Are they true then? The answer is *yes and no*. The corollary remains true without restriction, i.e.

2.4. Theorem. *If $f\colon V \to Y$ is a continuous map as above and* Fix (f^n) *is compact, then* $I_n(f) \equiv 0 \bmod n$. (For p prime cfr. [**13**] or [**10**]; for general n cfr. [**5**], th. 1.1).

The lemma itself can go wrong in geometric fixed point theory for various reasons. However

2.5 Theorem ([**5**], 6.2). *If n is odd and all fixed points of f^n are regular (= transversal), then $I_n(f)$ equals the number of points of period exactly n, each point counted with its multiplicity ± 1.*

If n is even the same equality is false, in general. It is to be corrected by an error term $-2r$, where r is the number of so-called «inverting» fixed points of $f^{n/2}$; these points have period exactly $n/2$ and they occur in full f-orbits (of length $n/2$) so that the error term is a multiple of n.

A typical example (with $n = 2$) is $f\colon \boldsymbol{R} \to \boldsymbol{R}$, $f(x) = -2x$. It has no point of period exactly $n = 2$, but $I_2(f) = -2$. It has one inverting fixed point.

3 - Periodic point rings

A good way to explain the relation between geometric and combinatorial fixed point theory is to introduce *Grothendieck rings*, both geometrically and combinatorially, and to compare them. For instance, in order to study points of period n we consider continuous maps $f\colon V \to Y$ as above (Y an ENR, $V \subset Y$ an open subset) such that $\operatorname{Fix}(f^n)$ is compact. Two such maps f_0, f_1 are said to be *n-equivalent* ($\underset{n}{\sim}$) if there is a third such map $F\colon W \to Z$ which lies over $[0, 1]$ (i.e. $p\colon Z \to [0, 1]$ is an $\mathrm{ENR}_{[0,1]}$ in the sense of [**4**], and $pF = p\,|\,W$) whose parts over 0 resp. 1 are homeomorphic to f_0 resp. f_1. Let $\mathfrak{P}_n$ denote the set of equivalence classes $[f]$ of such f. Geometric addition (topological sum) and multiplication (cartesian product) are compatible with n-equivalence; they define a commutative ring structure in $\mathfrak{P}_n$ with $0 = [\emptyset]$, $1 = [\mathrm{id}_{\mathrm{point}}]$, $-1 = [\boldsymbol{R} \xrightarrow{2} \boldsymbol{R}]$.

For every natural number k which divides n the index $I(f^k)$ is compatible with n-equivalence, i.e. we can define a homomorphism $\mathfrak{P}_n \to \boldsymbol{Z}$ by $[f] \mapsto I(f^k)$. Or we can use the Möbius functions (2.1) to define

$$\mu_k\colon \mathfrak{P}_n \to \boldsymbol{Z}\,, \qquad \mu_k[f] = I_k(f)\,, \tag{3.1}$$

a homomorphism of abelian groups. Conversely, we can define

$$i_k\colon \boldsymbol{Z} \to \mathfrak{P}_n\,, \qquad i_k(r) = r[\zeta_k]\,, \tag{3.2}$$

where $\zeta_k\colon Z_k \to Z_k$ is a k-cycle (thus Z_k is a set of k elements and ζ_k is a cyclic permutation). The Lemma 2.2 then shows that $\mu_k i_h = \delta_{kh} =$ Kronecker symbol. In other words, if we assemble the maps $\{\mu_k\}$ into a single homomorphism $\mu = \{\mu_k\}_{k/n}\colon \mathfrak{P}_n \to \boldsymbol{Z}^{d(n)}$, and the homomorphisms $\{i_k\}$ into a single homomorphism $i = \{i_k\}_{k/d}\colon \boldsymbol{Z}^{d(n)} \to \mathfrak{P}_n$, where $d(n) = |\{k \in N\,|\,k/n\}|$, then μi

is the identity map of $Z^{d(n)}$. Furthermore, the proof of 1.1 in [**5**] essentially shows that i is surjective, hence

3.3. Theorem. *$i\colon Z^{d(n)} \to \mathfrak{P}_n$ is isomorphic,* i.e. *$\mathfrak{P}_n$ is freely generated, as an additive group, by* $\{[\zeta_k]\}_{k/n}$.

A *variation of* $\mathfrak{P}_n$ imposes itself if we study points of period $\leqslant n$. In this case we consider maps $f\colon V \to Y$ (Y an ENR, $V \to Y$ open) such that Fix (f^ν) is compact for all ν with $1 \leqslant \nu \leqslant n$. Two such maps f_0, f_1 are said to be *equivalent up to n* $\left(\underset{(n)}{\sim}\right)$ if they can be connected (as above) by a third such map F over $[0, 1]$. Let $\mathfrak{P}_{(n)}$, denote the set of equivalence classes $[f]$ of such f; it is again a commutative ring as above. We obtain reciprocal (additive) isomorphisms

$$Z^n \xrightarrow{i} \mathfrak{P}_{(n)} \xrightarrow{\mu} Z^n\,, \tag{3.4}$$

as above, $i = \{i_\nu\}_{1\leqslant\nu\leqslant n}$, $\mu = \{\mu_\nu\}_{1\leqslant\nu\leqslant n}$. Thus

3.5 Theorem. *$\mathfrak{P}_{(n)}$ is freely generated, as an additive group by* $\{[\zeta_\nu]\}_{1\leqslant\nu\leqslant n}$.

Intuitively speaking, the isomorphism $\mu = i^{-1}\colon \mathfrak{P}_{(n)} \to Z^n$ when applied to an element $[f] \in \mathfrak{P}_{(n)}$ tells you how many ν-cycles occur in Fix (f^ν), for every $\nu = 1, 2, \ldots, n$. This count, however, should be taken with several grains of salt as the remarks before and after 2.5 indicate.

4 - Iterated maps and symmetric powers

Let $f\colon V \to Y$ denote a continuous map and f^k its iterates as above (in **2**), $k = 1, 2, \ldots$. We want to compare the indices $\{I(f^k)\}$ of the iterates with the indices of the symmetric power maps $\{\mathrm{SP}^k f\colon \mathrm{SP}^k V \to \mathrm{SP}^k Y\}_{k=1,2,\ldots}$.

4.1 Theorem. *For every $n \geqslant 1$, if* Fix $(\mathrm{SP}^n f)$ *is compact then so is* Fix (f^n). *Conversely, if* Fix (f^k) *is compact for all $k \leqslant n$, then* Fix $(\mathrm{SP}^n f)$ *is compact, and the index* $I(\mathrm{SP}^n f)$ *coincides with the coefficient of t^n in the (formal) power series* $\exp\left(\sum_{k=1}^{n} (I(f^k)/k)\,t^k\right)$. *In particular, if* Fix (f^k) *is compact for all $k = 1, 2, \ldots$, then*

$$\sum_{k=0}^{\infty} I(\mathrm{SP}^k f)\,t^k = \exp\Big(\sum_{k=1}^{\infty} \frac{I(f^k)}{k}\,t^k\Big)\,. \tag{4.2}$$

For instance, if Y is compact and f is the identity map of Y (hence

$I(f) = e(Y) =$ Euler characteristic of Y) then (4.2) becomes $\sum_k e(\mathrm{SP}^k Y)t^k = (1-t)^{-e(Y)}$, as asserted before (1.4).

Proof. If $K = \mathrm{Fix}\,(\mathrm{SP}^n f)$ is compact then so is $\pi^{-1}K$, where $\pi\colon V^n \to \mathrm{SP}^n V = V^n/S(n)$ is the projection map. On the other hand, the map

$$(4.3) \qquad (x_1, x_2, \ldots, x_n) \mapsto (fx_n, fx_1, \ldots, fx_{n-1})$$

maps $\pi^{-1}K$ into itself and has fixed point set homeomorphic to $\mathrm{Fix}\,(f^n)$ (via $(x_1, \ldots, x_n) \mapsto x_1$), hence $\mathrm{Fix}\,(f^n)$ is homeomorphic to a closed subset of $\pi^{-1}K$, and therefore compact.

Suppose now $\mathrm{Fix}\,(f^k)$ is compact for $1 \leqslant k \leqslant n$, hence $F = \bigcup_{k=1}^{n} \mathrm{Fix}\,(f^k)$ is compact; also $f(F) \subset F$. It follows that $\mathrm{SP}^n F$ is compact and $(\mathrm{SP}^n f)(\mathrm{SP}^n F) \subset \mathrm{SP}^n F$.

If $[x_1, \ldots, x_n]$ is a fixed point of $\mathrm{SP}^n f$ then the set $\{x_i\}_{1 \leqslant i \leqslant n}$ is mapped into itself by f, hence $x_i \in F$, hence $[x_1, \ldots, x_n] \in \mathrm{Sp}^n F$. Therefore $\mathrm{Fix}\,(\mathrm{SP}^n f) = \mathrm{Fix}\,(\mathrm{SP}^n f | \mathrm{SP}^n F)$ is compact.

This proves the first part of the theorem. For the second part we remark that $\alpha(f) = \sum_{k=0}^{n} I(\mathrm{SP}^k f)\, t^k$ as well as $\beta(f) = \exp\big(\sum_{k=1}^{n} (I(f^k)/k)\, t^k\big)$ and its n-th degree approximation are elements of $1 + t\boldsymbol{Z}[[t]]$, the multiplicative group of unital formal power series over $\boldsymbol{Z}$. More adequately, we should view them as elements of the quotient group $1 + t\boldsymbol{Z}[[t]]/(t^{n+1})$ i.e. calculate modulo terms of degree $\geqslant n+1$. The theorem then asserts $\alpha(f) = \beta(f)$. Both expressions are invariant under (n)-equivalence, i.e. both are functions of the equivalence class $[f] \in \mathfrak{P}_{(n)}$ (cfr. **3** above). Furthermore, both α and β are homomorphic, i.e. $\alpha(f \oplus g) = \alpha(f)\alpha(g)$, $\beta(f \oplus g) = \beta(f)\beta(g)$, where $\oplus$ denotes disjoint union (this easily follows from the additivity and multiplicativity of the index). In other words, α and β are homomorphisms

$$(4.4) \qquad \alpha, \beta\colon \mathfrak{P}_{(n)} \to 1 + t\boldsymbol{Z}[[t]]/(t^{n+1}) \,.$$

But $\mathfrak{P}_{(n)}$ is generated (cfr. Theorem 3.5) by the (classes of the) j-cycles $\zeta_j\colon Z_j \to Z_j$ for $j = 1, 2, \ldots, n$. In order to prove $\alpha = \beta$ it suffices therefore to show that $\alpha(\zeta_j) = \beta(\zeta_j)$. A point of $\mathrm{SP}^k Z_j$ is an (unordered) k-tuple in Z_j. If it is a fixed point of $\mathrm{SP}^k \zeta_j$ then it must contain every $z \in Z_j$ equally often. Therefore, $\mathrm{SP}^k \zeta_j$ has one fixed point if j divides k, and no fixed point otherwise, hence

$$\alpha(\zeta_j) = \sum_r t^{rj}\,, \qquad \text{where } 0 \leqslant rj \leqslant n\,.$$

Similarly, the iterate $(\zeta_j)^k$ has j fixed points if j divides k, and no fixed point otherwise. Hence

$$\beta(\zeta_j) = \exp\left(\sum_s \frac{t^{sj}}{s}\right) = \exp\left(-\log(1-t^j)\right) = \frac{1}{1-t^j} = \sum_r t^{rj}.$$

Therefore $\alpha(\zeta_j) \equiv \beta(\zeta_j)$ modulo t^{n+1}, as asserted.

5 - Indices of equivariant maps

Here we consider maps $f\colon V \to Y$ as above on which a finite group G of symmetries operates. More precisely, we start with a fixed finite group G, and we assume Y to be a G-ENR, $V \subset Y$ an open G-subset and f a continuous G-map with compact fixed point set $\mathrm{Fix}(f)$. For instance, every finite-dimensional $\boldsymbol{R}$-linear representation space ($\boldsymbol{R}G$-module) M is a G-ENR. More generally, every G-neighborhood retract of M is a G-ENR, i.e. every subspace Y which is the image of an idempotent G-map $\varrho\colon W \to W$, where $W \subset M$ is an open G-subset. Every smooth G-manifold Y (i.e. smooth manifold on which G acts by diffeomorphisms) is a G-ENR. Some references on G-ENRs (resp. G-ANRs) are [2], [7], [8], [11].

Two maps f_0, f_1 as above are said to be *G-equivalent* ($\sim_G$) if there is a third such map $F\colon W \to Z$ which lies over the interval $[0, 1]$ (i.e. $p\colon Z \to [0, 1]$ is a G-$\mathrm{ENR}_{[0,1]}$, $pF = p \,|\, W$), whose parts over 0 resp. 1 are G-homeomorphic to f_0 resp. f_1). Geometric G-fixed point theory deals with properties of f which are invariant under G-equivalence. To do it in a systematic way we introduce the Grothendieck-ring G-FIX in analogy to **3**: The elements of G-FIX are G-equivalence classes $[f]$ of G-maps $f\colon V \to Y$ as above. They are added and multiplied geometrically, i.e. by taking the topological sum $f_1 \oplus f_2$ resp. cartesian product $f_1 \times f_2$ of representative maps. This turns G-FIX into a commutative ring, with $0 = [\emptyset]$, $1 = [\mathrm{id}_{\mathrm{point}}]$, $-1 = [\boldsymbol{R} \xrightarrow{2} \boldsymbol{R}]$. The problem of G-invariants then becomes the problem of knowing the ring G-FIX. Fortunately, this can be reduced again to finite sets, as follows.

For every subgroup $H \subset G$ the group G operates transitively on the set G/H of (left) cosets of H. Let ι_H denote the identity map of G/H. Then

5.1 Theorem. *G-FIX is freely generated, as an additive group, by $\{[\iota_H]\}_{H \in \mathrm{Konj}(G)}$, where H runs through a complete system of pairwise non-conjugate subgroups of G.*

A better formulation of this result is in terms of the Burnside ring $A(G)$ which is the Grothendieck ring of finite G-sets. Every element a of $A(G)$ is a formal difference, $a = S - T$, of two finite G-sets S, T; and $S - T = S' - T'$ iff the disjoint unions $S \oplus T'$, $S' \oplus T$ are isomorphic G-sets. Addition and multiplication in $A(G)$ are quite obvious. Then

5.2 Theorem. *The map*

$$\iota\colon A(G) \to G\text{-FIX}\,, \qquad \iota(S - T) = [\mathrm{id}_S] - [\mathrm{id}_T]$$

is an isomorphism of rings.

This result is essentially due to tom Dieck (cfr. [**2**]). He proves $A(G) \cong \omega_G^0 = o$-th stable G-homotopy of S^0, while $G\text{-FIX} \cong \omega_G^0$ follows as in [**3**]. The book of tom Dieck also contains more information about G-ENRs and a wealth of results on equivariant homotopy theory. Theorem 5.2 can also be proved directly, without reference to equivariant homotopy theory. A direct and reasonably simple proof can be found in [**11**]. The main point is, of course, to show that ι is surjective.

6 - Parametrised fixed point theory and (combinatorics?)

Parametrised fixed point theory is an important extension of ordinary (geometric) fixed point theory. Roughly speaking, it consists in studying not one map $f\colon V \to Y$ as above but a continuous family $\{f_b\colon V_b \to Y_b\}$, $b \in B$, of such maps depending on a parameter b which varies in a topological space B. More conveniently (but less suggestive, perhaps), this is formulated in terms of spaces and maps over B. Thus $p\colon Y \to B$ is a map $\big(Y_b = p^{-1}(b)\big)$, $V \subset Y$ an open subset $(V_b = V \cap Y_b)$, $f\colon V \to Y$ is a map such that $pf = p|V$ $(f(V_b) \subset Y_b$, hence $f_b\colon V_b \to Y_b$ by restriction), and $p|\mathrm{Fix}(f)\colon \mathrm{Fix}(f) \to B$ is assumed to be *proper* ($\mathrm{Fix}(f_b)$ compact, and $p|\mathrm{Fix}(f)$ closed). Moreover, $p\colon Y \to B$ should be an ENR_B in the sense of [**4**] (Y_b an ENR with euclidean neighborhood retraction depending continuously on $b \in B$).

Two such maps f_0, f_1 (over the same B) are said to be *equivalent over* B $(\sim_B)$ if there is a third such map $F\colon W \to Z$ over $B \times [0, 1]$ whose parts over $B \times \{0\}$ resp. $B \times \{1\}$ are homeomorphic (over B) to f_0 resp. f_1. One can assemble the equivalence classes over B into a Grothendieck-ring

$$\mathrm{FIX}_B = \{f\}/\sim_B\,, \tag{6.1}$$

as above (e.g. **5**) and one can prove [**4**] that FIX_B is isomorphic to $\pi^0_{st}(B^+)$, the o-th stable cohomotopy of $B \oplus$ a point. In fact, one of the main attractions of parametrised fixed point theory is that (via pullbacks and fixed point transfers) it can be made into a category which is equivalent to the stable homotopy category (or stable shape category if *bad* spaces B are admitted). This is the main result of B. Schäfer in his thesis [**9**].

Clearly it would be very desirable then to connect and perhaps reduce parametrised fixed point theory to *parametrised combinatorics*. Perhaps the main and anyway the first problem is what *parametrised combinatorics* should be. An attractive candidate is the theory of (not necessarily connected) covering spaces. With Greame Segal's famous conjecture (1) about $\pi^0_{st}(BG^+)$, at the latest, it has become clear that the innocent-looking theory of covering spaces has an amazing potential in homotopy theory. Still, it would seem rather optimistic to expect the same simple answers from it for parametrised homotopy theory as in the unparametrised case $B = $ a point. As a test case one might consider the indices $I(f^k) \in \pi^0_{st}(B^+)$ of iterated parametrised maps f over B. What is the generalization of 2.4 in the parametrised case? I worked on this question for a while and gained the impression that covering spaces did not lead to a satisfactory answer here, but the impression isn't really well-founded; I didn't try hard enough.

Parametrised combinatorics probably should be a certain theory of finite to-one continuous maps, but not all such maps, and perhaps with more structural data?

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