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Unities, semantics and realizations (**)

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Let $\boldsymbol{P} = (P, \leqslant)$ be a poset with greatest and least elements, denoted by $\underline{1}$ and $\underline{0}$, respectively. Let A be a non-empty set, then

Def. 1. An ordered triple $\boldsymbol{M} = (\boldsymbol{P}, \models, =\!|)$ is a *pre-model for* A, if $\models$ and $=\!|$ are contained in $P \times A$ and for every $p, q \in P$, for every $a \in A$, the following hold: (1) $\underline{0} \models a$; (2) $\underline{1} =\!| a$; (3) if $p \leqslant q$ and $q \models a$, then $p \models a$; (4) if $p \leqslant q$ and $p =\!| a$, then $q =\!| a$; (5) if $p \models a$ and $q =\!| a$, then $p \leqslant q$.

Remark 1. Conditions (1)-(5) are similar to usual conditions on an ordering relation. Conditions (3)-(5) yield, for every p in P and for every a, b in A:

(6) if $p =\!| a$ and $p \models b$, then $(\forall q \in P)$ $(q \models a \Rightarrow q \models b)$; (7) if $p =\!| a$ and $p \models b$, then $(\forall q \in P)$ $(q =\!| b \Rightarrow q =\!| a)$.

Conditions (1)-(3) agree with definition of μ-pre-model for a set A, as given in [2].

An interesting comparison can be made with notion of unity, given in [1].

Proposition 1. *Let I be a unity of sets P and A, then there is a poset $\boldsymbol{P}'$ and relations* $\models, =\!| \leqslant P' \times A$, *such that* (3)-(5) *hold.*

Proof. As a consequence of Theorem 2 of [1], there is a preorder relation on P, given by $I \nearrow (P \times P)$, restriction of I to $P \times P$, and a relation $I \nearrow (P \times A)$,

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such that the given unity I is representable as a unity of the relation $I \nearrow (P\times A)$. Let $\boldsymbol{P}'=(P/\simeq, \leqslant)$ be the poset obtained as quotient $P/\simeq$, where $\simeq$ is the equivalence relation generated by $I \nearrow (P\times P)$. Denote by $\models\subseteq P'\times A$ and $\dashv^{op}$, respectively, the relations induced by $I \nearrow (P\times A)$ and $I \nearrow (A\times P)$. A straightforward calculation shows that (3)-(5) hold.

Remark 2. Let $\boldsymbol{P}=(P, \leqslant)$ be a poset and let A be a non-empty set. Given a unity I of sets P and A, it can happen that (pre)-order relation $I \nearrow (P\times P)$ be different from $\leqslant$. To avoid these difficulties it seems more suitable the following

Def. 2. (a) Let $\boldsymbol{P}=(P, \leqslant)$ be a poset (or a preordered set) and let $\models$ be a subset of $P\times A$, where A is a non-empty set. A *bounded unity of relations* $\leqslant$ *and* $\models$ is the largest preorder I on $P\oplus A$ (formally-disjoint union) whose restrictions to $P\times P$ and $P\times A$ are equal to $\leqslant$ and $\models$, respectively. (b) A *bounded unity on* $\boldsymbol{P}$ *and* A is any preorder I on $P\oplus A$ such that $I\nearrow(P\times P)=\leqslant$ and (8) $(\forall x\in P\oplus A)\,(\forall y\in A)\,(yIx \Leftrightarrow (\forall z\in P)\,(zIy \Rightarrow zIx))$.

With these definitions, the following hold

Proposition 2. *Given a poset (or a preordered set)* $\boldsymbol{P}=(P, \leqslant)$ *and a relation* $\models\subseteq P\times A$, *there is a largest preorder* I *on* $P\oplus A$ *such that* $I \nearrow (P\times P)=\leqslant$ and $I \nearrow (P\times A)=\models$ *if and only if* $\leqslant$ *and* $\models$ *satisfy condition* (3). *In this case for every* $p\in P$ *and for every* a, b *in* A: (9) aIp *if and only if* $(\forall q\in P)$ $(q\models a \Rightarrow q\leqslant p)$; (10) aIb *if and only if* $(\forall q\in P)\,(q\models a \Rightarrow q\models b)$.

Proof. It is easy to show that if exists a preorder I on $P\oplus A$ whose restrictions to suitable sets are given by $\leqslant$ and $\models$, respectively, condition (3) stating an instance of transitivity, must hold. Conversely if condition (3) holds, (9) and (10) complete the definition of a preorder on $P\oplus A$: reflexivity is obvious; repeated applications of (3), (9) and (10) allow us to conclude that I is transitive. Let J be a preorder on $P\oplus A$ such that $J \nearrow (P\times P)=\leqslant$ and $J \nearrow (P\times A)=\models$. Suppose $J\not\subseteq I$, then there are $p\in P$, $a, b\in A$ such that either aJp and not aIq or aJb and not aIb. In first case, by (9), there is $q\in P$ with $q\models a$ and $q\nleqslant p$. Hence qJa and aJp, but not qJp. In second case, by (10), there is $q\in P$ such that $q\models a$ and $q\not\models b$; hence qJa, aJb and not qJb.

Def. 2 (b) is related with the following

Proposition 3. *Let* $\boldsymbol{P}=(P, \leqslant)$ *be a poset (or a preordered set) and let*

A be a non-empty set. Any bounded unity on $\boldsymbol{P}$ and A is representable as a bounded unity of a pair of relations.

Proof. Let I be a bounded unity on $\boldsymbol{P}$ and A, denote $I \nearrow (P\times P)$ by $\leqslant$ and $I \nearrow (P\times A)$ by $\models$. These relations are restriction to suitable sets of a transitive relation on $P\oplus A$, then (3) holds. By Proposition 2 there is a largest preorder J on $P\oplus A$ whose restrictions are $\leqslant$ and $\models$, respectively. It follows that $I\subseteq J$. Let $a\in A$ and $p\in P$ be such that aJp; by (9), $(\forall q\in P)$ $(q\models a \Rightarrow q\leqslant p)$. This condition can be written as follows: $(\forall q\in P)(qIa \Rightarrow qIp)$. By (8) aIp. In a similar way, if $a, b\in A$ are such that aJb, then by (8), aIb. Hence $I = J$.

Remark 3. Let $\boldsymbol{M} = (\boldsymbol{P}, \models, =\!\!|)$ be a pre-model for A there is a bounded unity on $\boldsymbol{P}$ and A, naturally associated with the given pre-model. Condition (9) implies (5); it follows also that for every $p\in P$, $a\in A$, if $p =\!\!| \, a$, then aIp. Conversely, given a bounded unity I on $\boldsymbol{P}$ and A, relations $I\nearrow (P\times A)$ and $(I\nearrow (A\times P))^{op}$, trivially satisfy conditions (3)-(5), while condition (1) (and (2)) holds if and only if $\boldsymbol{P}$ has least (greatest) element and for every $a\in A$ there is $p\in P$ such that pIa (aIp). In this case $(\boldsymbol{P}, I\nearrow (P\times A), (I\nearrow (A\times P))^{op})$ is a pre-model for A.

We define now some notions related to pre-models.

Notation 1. (a) Let Q be a subset of P, denote by ΔQ (∇Q) the set of lower (upper) bounds of Q in $\boldsymbol{P}$. (b) Let $\boldsymbol{M} = (\boldsymbol{P}, \models, =\!\!|)$ be a pre-model for A. For any $a\in A$ set $M_a = \{p\in P\,|\,p\models a\}$ and $W_a = \{p\in P\,|\,p =\!\!| \, a\}$.

Remark 4. Given a pre-model $\boldsymbol{M}$ for A, for every $a\in A$, condition (5) implies that $M_a\subseteq \Delta W_a$ and $W_a\subseteq \nabla M_a$. Moreover $M_a\cap W_a$ is empty or is a singleton: if $p, q\in M_a\cap W_a$, then $p\leqslant q$ and $q\leqslant p$, by (5). In this case it is easy to prove that $p = \max M_a = \min W_a$. By condition (3) M_a is a downward closed subset of P and for (4) W_a is an upward closed subset of P.

Notation 2. Let $\boldsymbol{M} = (\boldsymbol{P}, \models, =\!\!|)$ be a pre-model for A, set:

(a) $N(\boldsymbol{M}) = \{a\in A\,|\,M_a = P\}$, $Z(\boldsymbol{M}) = \{a\in A\,|\, W_a = P\}$,
$T(\boldsymbol{M}) = N(\boldsymbol{M})\cup Z(\boldsymbol{M})$;

(b) $R(\boldsymbol{M}) = \{a\in A\,|\,M_a\cap W_a\neq\emptyset\}$, the set of formulae realizable in $\boldsymbol{M}$;

(c) $\boldsymbol{M}(A)^+ = \{a\in A\,|\,\Delta M_a = W_a\}$, $\boldsymbol{M}(A)^- = \{a\in A\,|\,\nabla W_a = M_a\}$.

In the sequel Notation 2 (c) would be simplified as $\boldsymbol{M}^+$ and $\boldsymbol{M}^-$.

Remark 5. Obviously $\underline{1} \models a$ if and only if $a \in N(\boldsymbol{M})$ and $\underline{0} \Dashv a$ if and only if $a \in Z(\boldsymbol{M})$.

Compare Notation 2 (a) and 2 (b) with Notations 3 and 4 of [2], respectively. Easily can be proved that a is realizable in $\boldsymbol{M}$ if and only if there is a unique $p \in P$ such that for every $q \in P$

$$(11) \qquad q \models a \text{ if and only if } q \leqslant p \quad \text{and} \quad q \Dashv a \text{ if and only if } p \leqslant q .$$

The set $T(\boldsymbol{M})$ is contained in $R(\boldsymbol{M})$, since $\boldsymbol{P}$ has least and greatest elements. Moreover by (11) it follows that $R(\boldsymbol{M})$ is contained in $\boldsymbol{M}^+ \cap \boldsymbol{M}^-$.

Def. 3. Let $\boldsymbol{L}$ be the propositional language (with truth symbol 1) and let $\boldsymbol{F}(A)$ be the set of formulae obtained taking A as set of propositional letters. An ordered triple $\boldsymbol{M} = (\boldsymbol{P}, \models, \Dashv)$ is *a model for* $\boldsymbol{F}(A)$ if

$$(12) \qquad \boldsymbol{M} \text{ is a pre-model for } \boldsymbol{F}(A) ,$$

$$(13) \qquad (\boldsymbol{P}, \models) \text{ is a } \nu\iota\eta\text{-model for } \boldsymbol{F}(A) \quad (\text{cf. [2]}) ;$$

or every $\alpha, \beta \in \boldsymbol{F}(A)$ and for every $p \in P$:

$$(14) \qquad p \Dashv (-\alpha) \text{ if and only if } (\forall q \in P)\,\big(q \models (-\alpha) \Rightarrow q \leqslant p\big) ,$$

$$(15) \qquad p \Dashv (\alpha \rightarrow \beta) \text{ if and only if } (\forall q \in P)\,\big(q \models (\alpha \rightarrow \beta) \Rightarrow q \leqslant p\big) ,$$

$$(16) \qquad p \Dashv (\alpha \wedge \beta) \text{ if and only if } (\forall q \in P)\,\big(q \models (\alpha \wedge \beta) \Rightarrow q \leqslant p\big) ,$$

$$(17) \qquad p \Dashv (\alpha \vee \beta) \text{ if and only if } (p \Dashv \alpha \text{ and } p \Dashv \beta) ,$$

$$(18) \qquad p \models (\alpha \vee \beta) \text{ if and only if } (\forall q \in P)\,\big(q \Dashv (\alpha \vee \beta) \Rightarrow p \leqslant q\big) .$$

Remark 6. Using previously introduced notations, Def. 3 can be restated as follows. $\boldsymbol{M}$ is a model for $\boldsymbol{F}(A)$ if the following hold: (12), (13) and

$$(14)' \quad W_{(-\alpha)} = \nabla M_{(-\alpha)}; \qquad (15)' \quad W_{(\alpha \rightarrow \beta)} = \nabla M_{(\alpha \rightarrow \beta)}; \qquad (16)' \quad W_{(\alpha \wedge \beta)} = \nabla M_{(\alpha \wedge \beta)};$$

$$(17)' \quad W_{(\alpha \vee \beta)} = W_\alpha \cap W_\beta; \qquad (18)' \quad M_{(\alpha \vee \beta)} = \Delta\, W_{(\alpha \vee \beta)} .$$

As a consequence, it follows that $(-\alpha)$, $(\alpha \rightarrow \beta)$ and $(\alpha \wedge \beta)$ belong to $\boldsymbol{M}^+$, $(\alpha \vee \beta)$ belongs to $\boldsymbol{M}^-$.

Remark 7. Definitions of morphisms of models and strong morphisms of models are similar to same definitions in [2], with the obvious clauses on relation $\dashv$. Let $f\colon \boldsymbol{M} \to \boldsymbol{M}'$ be a morphism of models, then $f(\underline{0}) = \underline{0}'$ and $f(\underline{1}) = \underline{1}'$, by Remarks 5 and 4. It follows that $N(\boldsymbol{M}) \subseteq N(\boldsymbol{M}')$ and $Z(\boldsymbol{M}) \subseteq Z(\boldsymbol{M}')$. If f is strong, then $N(\boldsymbol{M}) = N(\boldsymbol{M}')$ and $Z(\boldsymbol{M}) = Z(\boldsymbol{M}')$.

For every $\alpha, \beta \in \boldsymbol{F}(A)$, $M_\alpha \subseteq M_\beta$ if and only if $(\alpha \to \beta) \in N(\boldsymbol{M})$, then $M'_\alpha \subseteq M'_\beta$. An analogous condition does not hold for set W_α, W_β, even if f is a strong morphism of models; nevertheless if $W'_\alpha \subseteq W'_\beta$, then $W_\alpha \subseteq W_\beta$. To correct this state of affairs, we give the following

Def. 4. Let $f\colon \boldsymbol{M} \to \boldsymbol{M}'$ be a morphism of models; f is *rigid* if f is strong and for every $p' \in P'$ and every formula α, $p' \not\dashv{}' \alpha$ implies that there is $p \in P$ such that $p' \leqslant' f(p)$ and $p \not\dashv \alpha$.

Proposition 4. *Let $f\colon \boldsymbol{M} \to \boldsymbol{M}'$ be a rigid morphism of models, then for every $\alpha, \beta \in \boldsymbol{F}(A)$, $W_\alpha \subseteq W_\beta$ if and only if $W'_\alpha \subseteq W'_\beta$.*

Proof. Suppose $W'_\alpha \not\subseteq W'_\beta$, the there is $p' \in W'_\alpha$ and $p' \notin W'_\beta$. By hypothesis there is $p \in P$ such that $p' \leqslant' f(p)$ and $p \notin W_\beta$. By (4), $f(p) \in W'_\alpha$ and also $p \in W_\alpha$. Hence $p \in W_\beta$, contradiction.

Theorem 5. *Let $\boldsymbol{M} = (\boldsymbol{P}, \models, \dashv)$ be a pre-model for A, then there is a «unique» extension of $\boldsymbol{M}$ to a model for $\boldsymbol{F}(A)$, with the same poset $\boldsymbol{P}$.*

Proof. Is obtained by induction.

For a model $\boldsymbol{M}$, the set $T(\boldsymbol{M})$ has «nice» properties, stated in the following

Proposition 6. *Let $\boldsymbol{M} = (\boldsymbol{P}, \models, \dashv)$ be a model for $\boldsymbol{F}(A)$, then for every formula α and β, the following hold:*

(a) *if $W_\alpha = P$, then $M_\alpha = \{\underline{0}\}$ and if $M_\alpha = P$, then $W_\alpha = \{\underline{1}\}$;*

(b) *if $\alpha \in \boldsymbol{M}^+$, then $M_\alpha = \{\underline{0}\}$ if and only if $W_\alpha = P$ and if $\alpha \in \boldsymbol{M}^-$, then $W_\alpha = \{\underline{1}\}$ if and only if $M_\alpha = P$;*

(c) *$(\alpha \wedge \beta) \in N(\boldsymbol{M})$ if and only if $\alpha, \beta \in N(\boldsymbol{M})$;*

(d) *if $\alpha, \beta \in T(\boldsymbol{M})$, then $(\alpha \wedge \beta) \in T(\boldsymbol{M})$;*

(e) *if $\alpha \in T(\boldsymbol{M})$, then $(-\alpha) \in T(\boldsymbol{M})$;*

(f) *if $\alpha, \beta \in T(\boldsymbol{M})$, then $(\alpha \to \beta) \in T(\boldsymbol{M})$;*

(g) *if $\alpha, \beta \in T(\boldsymbol{M})$, then $(\alpha \vee \beta) \in T(\boldsymbol{M})$;*

(h) *$\boldsymbol{F}(\emptyset) \subseteq T(\boldsymbol{M})$.*

Proof. (a) Trivial by Remark 4 and conditions (1) and (2). (b) Trivial, from (a) and Notation 2 (c). (c) Trivial by Notation 2 (a); (13) (and (4′) of [2]). (d) Let either α or β be element of $Z(\boldsymbol{M})$, then by (a) above, $M_{(\alpha\wedge\beta)} = \{\underline{0}\}$. By Remark 6, $(\alpha\wedge\beta) \in \boldsymbol{M}^+$, then by (b), $(\alpha\wedge\beta) \in Z(\boldsymbol{M})$. (e) If $\alpha \in N(\boldsymbol{M})$, $M_{(-\alpha)} = \{\underline{0}\}$, then, using Remark 6 and (b), $(-\alpha) \in Z(\boldsymbol{M})$. If α is an element of $Z(\boldsymbol{M})$, by (a), $M_\alpha = \{\underline{0}\}$, hence, by (2′) of [2], $M_{(-\alpha)} = P$, i.e. $(-\alpha) \in N(\boldsymbol{M})$. (f) Since $(\alpha \to \beta) \in \boldsymbol{M}^+$ (Remark 6), by (b), proof is the same as proof of Proposition 4 (d) of [2]. (g) If either α or β (or both), belongs to $N(\boldsymbol{M})$, then $(\alpha\vee\beta) \in N(\boldsymbol{M})$: by (a) $W_\alpha = \{\underline{1}\}$ or $W_\beta = \{\underline{1}\}$; by (17)′ $W_{(\alpha\vee\beta)} = \{\underline{1}\}$. Since, by Remark 6, $(\alpha\vee\beta) \in \boldsymbol{M}^-$, from (b) it follows that $(\alpha\vee\beta) \in N(\boldsymbol{M})$. In case $\alpha, \beta \in Z(\boldsymbol{M})$, by (17)′, $W_{(\alpha\vee\beta)} = P$, i.e. $(\alpha\vee\beta) \in Z(\boldsymbol{M})$. (h) From (13) (and (1′) or [2]), it follows that $1 \in N(\boldsymbol{M})$; (c)-(g) give the result for all formulae of $\boldsymbol{F}(\emptyset)$, built up out with truth symbol and connectives.

Notation 3. Given a formula α, denote by Sub (α) the set of all subformulae of α.

Def. 5. (a) Let $\boldsymbol{M}$ be a model for $\boldsymbol{F}(A)$, a formula α is *hereditarily realizable in* $\boldsymbol{M}$ if Sub $(\alpha) \subseteq R(\boldsymbol{M})$. Denote by $H(\boldsymbol{M})$ the set of hereditarily realizable formulae.

(b) A subset F of $\boldsymbol{F}(A)$ is a *fragment* if for every $\alpha \in F$, Sub $(\alpha) \subseteq F$.

Theorem 7. *There exists a poset* $\boldsymbol{P}$, *with least and greatest element, such that for every non-empty fragment* F *there are two relations* $\models, \mathrel{=\!|} \subseteq P \times \boldsymbol{F}(A)$ *such that* $\boldsymbol{M} = (\boldsymbol{P}, \models, \mathrel{=\!|})$ *is a model for* $\boldsymbol{F}(A)$ *in which* $H(\boldsymbol{M}) = \boldsymbol{F}(F \cap A) \subseteq T(\boldsymbol{M})$.

Proof. Proof is similar to proof of Theorem 5 of [2]. Take $\boldsymbol{P}$ as the set $\{\underline{0}, \underline{1}\}^2$, with lexicographical order induced by natural order on $\{\underline{0}, \underline{1}\}$. Trivially $(\underline{0}, \underline{0})$ is the least element and $(\underline{1}, \underline{1})$ is the greatest element of $\boldsymbol{P}$. Given a fragment F, define $\models_F$ as made in Theorem 5 of [2]. By Proposition 2, there is a bounded unity of relations $\leqslant$ and $\models_F$ on poset $\boldsymbol{P}$ and set A. By second part of Remark 3, in this way we get a pre-model $\boldsymbol{M}_F$ for A. Extend $\boldsymbol{M}_F$ to a model $\boldsymbol{M}$, for $\boldsymbol{F}(A)$, as stated in Theorem 5. Prove, by induction, that the set $\boldsymbol{F}(F \cap A)$ is contained in $T(\boldsymbol{M})$: $1 \in T(\boldsymbol{M})$ and $\alpha \in A$ is in $T(\boldsymbol{M})$ if and only if $\alpha \in F \cap A$. The remainder is a consequence of Proposition 6. Using induction, it can be proved that $H(\boldsymbol{M}) = \boldsymbol{F}(F \cap A)$: first step is trivial, i.e. $H(\boldsymbol{M}) \cap A = F \cap A$, by definition of the model $\boldsymbol{M}$. Suppose $\alpha, \beta \in \boldsymbol{F}(F \cap A)$ and $\alpha, \beta \in H(\boldsymbol{M})$, then $M_{(-\alpha)} \cap W_{(-\alpha)} = \{\underline{0}\}$ otherwise $M_{(-\alpha)} \cap W_{(-\alpha)} = \{\underline{1}\}$ hence, in both cases, $(-\alpha) \in R(\boldsymbol{M})$. In a similar way can be proved that $M_{(\alpha\otimes\beta)} \cap W_{(\alpha\otimes\beta)} \neq \emptyset$, for $\otimes \in \{\vee, \wedge, \to\}$. Then $\alpha \otimes \beta \in R(\boldsymbol{M})$. In conclusion

$F(F \cap A) \subseteq H(M)$. Conversely, if $\alpha \in H(M)$, then propositional letters in Sub (α) must be elements of $H(M) \cap A = R(M_F) = F \cap A$, or $\alpha \in F(\emptyset)$; it follows that $\alpha \in F(F \cap A)$. Hence $F(F \cap A) = H(M)$.

Given a model for $F(A)$, $M = (P, \models, \dashv)$, and a fragment F such that $H(M) \subseteq F \subseteq M^{+} \cap M^{-}$, we construct a new model $M[F]$ in which M can be rigidly embedded. To prove this, first consider the bounded unity of relations $\leqslant$ and $\models \nearrow (P \times F)$, and denote this by I_F. The relation I_F is a preorder on $P \oplus F$. Note that condition on the fragment F allows a simplification of (9): for $p \in P$ and $\varphi \in F$, $\varphi I_F p$ if and only if $p \dashv \varphi$.

Lemma 8. *Define* $\models(F)$, $\dashv(F) \subseteq (P \oplus F) \times F(A)$, *extending* $\models$ *and* $\dashv$ *of* M, *setting for every* $\varphi \in F$ *and every* $\alpha \in F(A)$, $\varphi \models (F)\alpha$ *if and only if* $M_\varphi \subseteq M_\alpha$ *and* $\varphi \dashv (F)\alpha$ *if and only if* $W_\varphi \subseteq W_\alpha$, *then* I_F *and* $\models(F)$, $\dashv(F)$ *satisfy conditions* (3)-(5).

Proof. Is a straightforward calculation.

Denote by $\simeq$ the equivalence relation generated by I_F on $P \oplus F$, and set $P[F] = P \oplus F/\simeq$. The unity I_F gives rise to an order $\leqslant[F]$ on $P[F]$; then set $P[F] = (P[F], \leqslant[F])$.

Lemma 9. *Let* p, q *be elements of* P *and* $\varphi, \psi \in F$, *then the following hold*:

(a) $p \simeq q$ *if and only if* $p = q$;

(b) $p \simeq \varphi$ *if and only if* $M_\varphi \cap W_\varphi = \{p\}$;

(c) $\varphi \simeq \psi$ *if and only if* $(\varphi \leftrightarrow \psi) \in N(M)$.

Proof. (a) Trivial. (b) If $p \simeq \varphi$, then $p I_F \varphi$, i.e. $p \models \varphi$ and $\varphi I_F p$, i.e. $p \dashv \varphi$. By Remark 4, $M_\varphi \cap W_\varphi = \{p\}$. (c) By (10) $\varphi I_F \psi$ if and only if $(\varphi \to \psi) \in N(M)$ and then by Proposition 6 (c), $\varphi \simeq \psi$ if and only if $(\varphi \leftrightarrow \psi) \in N(M)$.

Lemma 10. *Let* x, y *be elements of* $P \oplus F$, *such that* $x \simeq y$, *then for every* $\alpha \in F(A)$, $x \models (F)\alpha$ *if and only if* $y \models (F)\alpha$ *and* $x \dashv (F)\alpha$ *if and only if* $y \dashv (F)\alpha$.

Proof. Is trivial in case $x, y \in P$, by Lemma 9 (a). If $x \in P$ and $y \in F$, then $M_y \cap W_y = \{x\}$. If $x \models (F)\alpha$, then $x \models \alpha$ and $x \dashv y$. Hence, every $p \in M_y$ is such that $p \leqslant x$, by (5), and by (3), $p \in M_\alpha$; therefore $M_y \subseteq M_\alpha$. Conversely, if $y \models (F)\alpha$, $p \models \alpha$, being $p \in M_y$. Proof for $\dashv(F)$ is similar. In case $x, y \in F$, by Lemma 9 (c), $M_x = M_y$ and also $W_x = \nabla M_x = \nabla M_y = W_y$, since $F \subseteq M^{+}$. Result follows trivially.

Denote elements of $P[F]$, using square bracket notation: if $x \in P \oplus F$, $[x] \in P[F]$.

Lemma 11. *Let α be a formula and $x \in P \oplus F$, define $[x] \models [F]\alpha$ if and only if $x \models (F)\alpha$ and $[x] \Dashv [F]\alpha$ if and only if $x \Dashv (F)\alpha$. Then $\boldsymbol{M}[F] = (\boldsymbol{P}[F], \models [F], \Dashv [F])$ is a model for $\boldsymbol{F}(\varDelta)$.*

Proof. Trivially relations $\models [F]$ and $\Dashv [F]$ are contained in $\boldsymbol{P}[F] \times \boldsymbol{F}(\varDelta)$. Lemma 10 says that these relations are correctly defined. Observe that $[\underline{0}]$ and $[\underline{1}]$ are the least and the greatest elements of $\boldsymbol{P}[F]$, respectively; then conditions (1) and (2) are trivial. Lemma 8 states that also conditions (3)-(5) hold. In conclusion (12) is proved. Moreover (13) can be proved as in Theorem 6 of [**2**].

Take now $x \in P \oplus F$ and a formula α, if $[x] \Dashv [F](-\alpha)$, then $x \Dashv (F)(-\alpha)$; if $x \in P$, for every $p \in P$ such that $p \models (-\alpha)$, by (5), $[p] \leqslant [F][x]$; for every $\varphi \in F$ such that $\varphi \models (F)(-\alpha)$, $M_\varphi \subseteq M_{(-\alpha)}$. It follows $W_{(-\alpha)} = \nabla M_{(-\alpha)} \subseteq \nabla M_\varphi = W_\varphi$, then $x \in W_\varphi$, i.e. $[\varphi] \leqslant [F][x]$. In case $x \in F$, $x \Dashv (F)[(-\alpha)$ means $W_x \subseteq W_{(-\alpha)}$; then if $p \in M_{(-\alpha)}$, $p \in M_x$, i.e. $[p] \leqslant [F][x]$, since $M_{(-\alpha)} \subseteq \varDelta\, W_{(-\alpha)} \subseteq \varDelta W_x = M_x$. If $\varphi \in F$ is such that $[\varphi] \models [F](-\alpha)$, $M_\varphi \subseteq M_{(-\alpha)}$ and $M_{(-\alpha)} \subseteq M_x$, as before; then $M_\varphi \subseteq M_x$, that is $[\varphi] \leqslant [F][x]$. Conversely, if $[x] \not\Dashv [F](-\alpha)$, in case that $x \in P$, $x \not\Dashv (-\alpha)$ and then there is $q \in P$, $q \models (-\alpha)$ and $q \nleqslant x$; this can be restated saying that there is $[y]$ in $P[F]$ such that $[y] \models [F](-\alpha)$ and $[y] \nleqslant [F][x]$. In case that $x \in F$, $W_x \nsubseteq W_{(-\alpha)}$ and there is $q \in W_x$ and $q \notin W_{(-\alpha)}$; therefore there is $p \in M_{(-\alpha)}$ such that $p \nleqslant q$. But $[p] \models [F](-\alpha)$ and if $(\forall y \in P \oplus F)([y] \models [F](-\alpha) \Rightarrow [y] \leqslant [F][x])$, then $[p] \leqslant [F][x]$. By (5) we get $p \leqslant q$, contradiction. In conclusion, condition (14) is proved. In a similar way conditions (15) and (16) can be proved. To show (17), let x be an element of $P \oplus F$, and $[x] \Dashv [F](\alpha \vee \beta)$; when $x \in P$, $[x] \Dashv [F](\alpha \vee \beta)$ is equivalent to $x \Dashv \alpha$ and $x \Dashv \beta$, i.e. $[x] \Dashv [F]\alpha$ and $[x] \Dashv [F]\beta$. In case $x \in F$, by (17)$'$, the following are equivalent: $W_x \subseteq W_{(\alpha \vee \beta)}$ and $W_x \subseteq W_\alpha$, $W_x \subseteq W_\beta$. To prove (18), take $x \in P \oplus F$ such that $[x] \models [F](\alpha \vee \beta)$, then $x \models (F)(\alpha \vee \beta)$; if $x \in P$, then $x \models (\alpha \vee \beta)$ and for every $q \in W_{(\alpha \vee \beta)}$, $x \leqslant q$, therefore $[x] \leqslant [F][q]$. If $\varphi \in F$ and $\varphi \Dashv (F)(\alpha \vee \beta)$, then $W_\varphi \subseteq W_{(\alpha \vee \beta)}$, hence $M_{(\alpha \vee \beta)} = \varDelta\, W_{(\alpha \vee \beta)} \subseteq \varDelta\, W_\varphi = M_\varphi$; it follows that $x \in M_\varphi$, i.e. $[x] \leqslant [F][\varphi]$. In case $x \in F$, hypothesis means that $M_x \subseteq M_{(\alpha \vee \beta)}$; if $q \in W_{(\alpha \vee \beta)}$, $q \in W_x$, since $W_{(\alpha \vee \beta)} \subseteq \nabla M_{(\alpha \vee \beta)} \subseteq \nabla M_x = W_x$, then $[x] \leqslant [F][q]$; if $\varphi \in F$ is such that $\varphi \Dashv (F)(\alpha \vee \beta)$, then, as before, $M_x \subseteq M_{(\alpha \vee \beta)} \subseteq M_\varphi$, i.e. $[x] \leqslant [F][\varphi]$. Conversely let $x \in P \oplus F$ be such that for all $[y] \Dashv [F](\alpha \vee \beta)$, $[x] \leqslant [F][y]$ and suppose $[x] \not\models [F](\alpha \vee \beta)$. That means $x \not\models (F)(\alpha \vee \beta)$; in case $x \in P$, there is $q \in W_{(\alpha \vee \beta)}$ such that $p \nleqslant q$, i.e. $[p] \nleqslant [F][q]$, contrary to assumption. If $x \in F$, $M_x \nsubseteq M_{(\alpha \vee \beta)}$, then there is $p \in M_x$, with $p \notin M_{(\alpha \vee \beta)}$. It follows that there is $q \in M_{(\alpha \vee \beta)}$ such that $p \nleqslant q$. But $[x] \leqslant [F][q]$, then $q \in W_x$ and by (5), $p \leqslant q$, contradiction.

Some other properties of the model $\mathbf{M}[F]$ are shown in the following

Lemma 12. *Let* $i\colon P \to P[F]$, *be defined by* $i(p) = [p]$, *then* i *is an injection and a rigid morphism of models* $i\colon \mathbf{M} \to \mathbf{M}[F]$.

Proof. First part of proof is trivial: i is injective and is a strong morphism of models, by definition of $\mathbf{M}[F]$. Let $x \in P \oplus F$ be such that $[x] \not=\!| \ [F]\alpha$, where α is a formula; if $x \in P$, $[x] \leqslant [F][x] = i(x)$, and $x \not=\!| \ \alpha$. In case $x \in F$, $W_x \not\subseteq W_\alpha$, then there is $p \in W_x$ and $p \notin W_\alpha$; it means that $[x] \leqslant [F]i(p)$ and $p \not=\!| \ \alpha$.

Lemma 13. (a) $\mathbf{M}^+ = \mathbf{M}[F]^+$ *and* $\mathbf{M}^- = \mathbf{M}[F]^-$, *then* $\mathbf{M}^+ \cap \mathbf{M}^- = \mathbf{M}[F]^+ \cap \mathbf{M}[F]^-$. (b) $F \subseteq H(\mathbf{M}[F])$.

Proof. (a) $\mathbf{M}^+ \subseteq \mathbf{M}[F]^+$ since i is rigid: take $\alpha \in \mathbf{M}^+$ and suppose $\nabla M[F]_\alpha \neq W[F]_\alpha$, then there is $[x] \in \nabla M[F]_\alpha$ such that $[x] \not=\!| \ [F]\alpha$; by rigidity of i there is $p \in P$ such that $[x] \leqslant [F][p]$ and $p \not=\!| \ \alpha$. But $[p] \in \nabla M[F]_\alpha$, then $p \in \nabla M_\alpha$, i.e. $p \in W_\alpha$. $\mathbf{M}[F]^+$ is contained in $\mathbf{M}^+$: if $\alpha \in \mathbf{M}[F]^+$, let $p \in P$ be an element of ∇M_α, if $[p]$ is in $\nabla M[F]_\alpha$, then $[p] \in W[F]_\alpha$ and so $p \in W_\alpha$, since i is a strong morphism of models; otherwise there is $\varphi \in F$ such that $[\varphi] \models [F]\alpha$ and $[\varphi] \not\leqslant [F][p]$, i.e. $M_\varphi \leqslant M_\alpha$ and $p \notin W_\varphi$. But $\varphi \in \mathbf{M}^+$, thence there is $q \in M_\varphi$ such that $q \not\leqslant p$; by previous inclusion, $q \in M_\alpha$ and by (5), $q \leqslant p$, contradiction. In a similar way can be proved that $\mathbf{M}^- \subseteq \mathbf{M}[F]^-$, using a condition of «co-rigidity» on i (i.e. reversing relations $\leqslant$ and $\not=\!|$) and that $\mathbf{M}[F]^- \subseteq \mathbf{M}^-$. (b) It is enough to prove that $F \subseteq R(\mathbf{M}[F])$, since F is a fragment. Take $\varphi \in F$, trivially $[\varphi] \models [F]\varphi$ and $[\varphi] =\!| \ [F]\varphi$, hence $M[F]_\varphi \cap W[F]_\varphi \neq \emptyset$.

This conclude the proof of the assertion stated before Lemma 8. The model $\mathbf{M}[F]$ can be completely characterized by a property of «freedom». The following theorem resumes these results

Theorem 14. *Let* $\mathbf{M} = (\mathbf{P}, \models, =\!|)$ *be a model for* $\mathbf{F}(A)$ *and let* F *be a fragment such that* $H(\mathbf{M}) \subseteq F \subseteq \mathbf{M}^+ \cap \mathbf{M}^-$, *then there is a model* $\mathbf{M}[F]$, *for* $\mathbf{F}(A)$, *in which* $\mathbf{M}$ *can be rigidly embedded, by means of* $i\colon \mathbf{M} \to \mathbf{M}[F]$. *Moreover* $F \subseteq H(\mathbf{M}[F])$ *and* $\mathbf{M}^+ \cap \mathbf{M}^- = \mathbf{M}[F]^+ \cap \mathbf{M}[F]^-$.

For every model $\mathbf{M}' = (\mathbf{P}', \models', =\!|')$, *for* $\mathbf{F}(A)$, *such that* $F \subseteq H(\mathbf{M}')$ *and for every rigid morphism of models* $f\colon \mathbf{M} \to \mathbf{M}'$, *there is a unique rigid morphism of models* $f'\colon \mathbf{M}[F] \to \mathbf{M}'$, *such that* $f' \circ i = f$.

Proof. Let $\mathbf{M}'$ and f satisfy hypothesis, define $f'\colon P[F] \to P$ as follows $f'([p]) = f(p)$ and $f'([\varphi]) = \max M'_\varphi$. Note that if $p \simeq \varphi$, in $P \oplus F$, then $p \models \varphi$ and $p =\!| \ \varphi$, simultaneously; thence $f(p) \models' \varphi$ and $f(p) =\!|' \varphi$. Therefore $M'_\varphi \cap W'_\varphi \neq \emptyset$ and $f'([p]) = f(p) = f'([\varphi])$. If $\varphi \simeq \psi$, then by Lemma 9 (c),

$(\varphi \leftrightarrow \psi) \in N(\boldsymbol{M})$ and by Remark 7, $(\varphi \leftrightarrow \psi) \in N(\boldsymbol{M}')$; it follows that $M'_\varphi = M'_\psi$ and also $\max M'_\varphi = \max M'_\psi$, i.e. $f'([\varphi]) = f'([\psi])$. This shows that f' is correctly defined.

Let x, y be elements of $P \oplus F$. If $[x] \leqslant [F][y]$, we show, by cases, that $f'([x]) \leqslant' f'([y])$. When $x, y \in P$ is trivial. If $x \in P$ and $y \in F$, $[x] \leqslant [F][y]$ means $x \models y$, then $f(x) \models' y$. By hypothesis $F \subseteq H(\boldsymbol{M}')$, then $f'([x]) = f(x) \leqslant' \max M'_y = f'([y])$, by (11). In case $x \in F$ and $y \in P$, $[x] \leqslant [F][y]$ means $y \dashv x$, then $f(y) \dashv' x$; again by (11), $f'([x]) = \min W'_x \leqslant' f(y) = f'([y])$. When $x, y \in F$, $[x] \leqslant [F][y]$ means $M_x \subseteq M_y$, then by Remark 7, $M'_x \subseteq M'_y$, then $f'([x]) = \max M'_x \leqslant' \max M'_y = f'([y])$. We can easily reverse passages above and conclude $[x] \leqslant [F][y]$ if and only if $f'([x]) \leqslant' f'([y])$. Let x be an element of $P \oplus F$ and let α be a formula. Suppose $[x] \models [F]\alpha$. If $x \in P$, this is equivalent to $x \models \alpha$ and also $f(x) \models' \alpha$, since f is strong, thence $[x] \models [F]\alpha$ if and only if $f'([x]) \models' \alpha$. In case $x \in F$, $[x] \models [F]\alpha$ is equivalent to $M_x \subseteq M_\alpha$; this condition can be written: $(x \to \alpha) \in N(\boldsymbol{M})$. By Remark 7, $N(\boldsymbol{M}) = N(\boldsymbol{M}')$, then $[x] \models [F]\alpha$ is equivalent to $M'_x \subseteq M'_\alpha$. But $x \in H(\boldsymbol{M}')$ and so there is p' such that $M'_x \cap W'_x = \{p'\}$ and $p' = \max M'_x$, $p' = f'([x])$. By (11) and (3), $[x] \models [F]\alpha$ is equivalent to $f'([x]) \models' \alpha$.

Suppose now $[x] \dashv [F]\alpha$. If $x \in P$, this is equivalent to $x \dashv \alpha$ and also to $f(x) \dashv' \alpha$, since f is a strong morphism; thence $[x] \dashv [F]\alpha$ if and only if $f'([x]) \dashv' \alpha$. In case $x \in F$, $[x] \dashv [F]\alpha$ is equivalent to $W_x \subseteq W_\alpha$. But f is a rigid morphism of models and by Proposition 4, $W'_x \subseteq W'_\alpha$, then $f'([x]) = \min W'_x \dashv' \alpha$. Conversely if $f'([x]) \dashv' \alpha$, for every $p \in W_x$, $f'([x]) \leqslant' f(p)$, by (11). It follows that $f(p) \in W'_\alpha$, by (4), and $p \in W_\alpha$, since f is strong. In conclusion $W_x \subseteq W_\alpha$, i.e. $[x] \dashv [F]\alpha$. Therefore f' is a strong morphism of models. Suppose $p' \in P'$, $\alpha \in \boldsymbol{F}(A)$ are such that $p' \not\dashv' \alpha$, then there is $p \in P$ such that $p' \leqslant' f(p)$ and $p \not\dashv \alpha$. This can be written as: there is $[x] \in P[F]$ such that $p' \leqslant' f'([x])$ and $[x] \not\dashv [F]\alpha$. Thence f' is a rigid morphism of models. Trivially $f' \circ i = f$, by definition of f'. Uniqueness of f', satisfying condition $f' \circ i = f$, is trivial.

References

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Abstract

Recently H. Crapo (cf. [1]) introduced the concept of unity of a relation in the context of representation theory for finite lattices, suggesting an application to Logic. Starting from that, I give a slightly modified version of the notion of unity, in order to obtain an interesting semantics. In this field I extend the notion of realizability, given in [2], and prove the following result: For each model **M** *and fragment F of propositional logic, satisfying a few mild hypothesis, there is an extension* **M**[F] *in which every element of F is realizable.*

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