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A nonlocal phenomenological theory in superconductivity (**)

A LUIGI CAPRIOLI per il suo 70° compleanno

1 - Introduction

It is known that superconductive phenomena make their appearance in metals, in metallic conductors and in several compounds at temperatures near absolute zero [17].

Superconductivity was first discovered by K. Onnes [12] in 1911 in the course of an investigation of the electrical resistance of various metals at about 3° K. He observed the first characteristic property of a superconductor: that is, its electrical resistance, for all practical purposes, is zero below a well-defined temperature T_c called the « critical » or « transition » temperature.

At any temperature T below T_c, the application of a magnetic field $\boldsymbol{H}_c(T)$, or a suitable transport current producing it, destroys the superconductivity and restores the normal resistance appropriate to the field. $\boldsymbol{H}_c$ is called « thermodynamic critical field ».

In 1933 Meissner and Ochsenfeld [10] observed that the magnetic induction $\boldsymbol{B}$ inside a bulk superconductor vanishes. It is the so called « Meissner effect » which is to be regarded as the second foundamental characteristic of a superconductor. Further experimental results [17], $[13]_1$, [15] showed that $\boldsymbol{B}$

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takes nonzero values in a thin region near the boundary of the sample and drops to a vanishingly small value over a characteristic length called «field penetration depth».

It was emphasized by F. London [7] that the pure superconducting state in a magnetic field has a persistent shielding current associated with it. This current, called «supercurrent», is always determined by the local magnetic field; for sample of macroscopic size it must be confined to a region very close to the surface.

In order to give a consistent description of the electromagnetic properties of superconductors listed above Maxwell's equations have been combined with London's ones [**8**]$_{1,2}$

$$\text{(1)} \qquad \frac{\partial}{\partial t}(\Lambda \boldsymbol{J}) = \boldsymbol{E}, \qquad\qquad \text{(2)} \qquad \nabla(\Lambda \boldsymbol{J}) = -\mu \boldsymbol{H},$$

with $\Lambda = m/ne^2$, m, e and n being respectively the mass, charge and volume density of the conduction electrons. The former accounts for the zero-resistance phenomenon, the latter explains the Meissner effect.

Although the London theory provided a framework for organizing most of the experimental data, nervertheless it was not able to give an exhaustive explanation of all superconductive phenomena. It turned out that various anomalous results do not fit into the theoretical picture provided by (1) and (2): e.g. the variation of λ with orientation in a single crystal [**13**]$_2$ and the strong change observed in the penetration depth if impurities are added to pure materials in so small amount that all others physical properties remain unchanged [**13**]$_3$.

To describe these anomalous effects Pippard [**13**]$_3$ proposed the following nonlocal relation between the current density $\boldsymbol{J}$ at a point and the vector potential $\boldsymbol{A}$

$$\text{(3)} \qquad \boldsymbol{J}(x) = -\frac{3ne^2}{\xi_0 m} \int \frac{\boldsymbol{r}(\boldsymbol{A}(x')\cdot\boldsymbol{r})}{R^4} \exp[-R/\xi]\,\mathrm{d}v',$$

where $\boldsymbol{r} = x - x'$, $R = |\boldsymbol{r}|$ and $\nabla\times\boldsymbol{A} = \boldsymbol{H}$. The constant ξ_0 is a characteristic parameter for the pure material and ξ, called «coherence length», is related to ξ_0 and the mean free path l.

The two striking advantages that the Pippard theory made were the elucidation of the non-local nature of the current-field relation and the specification of the coherence length.

Unfortunately, Pippard's relation must be added to equations (1)-(2) not replacing them so that the nonlocal theory arising from (3) does not represent a new mathematical model of superconductivity but it may be considered merely as a modification of the London one. In essence, the empirical nature of equation (3) is due to the fact that Pippard obtained his results empirically.

Improving previous results [**5**] in the first part of this paper we make some remarks on the London theory. The former remark (sects. **3**, **4**) shows that equation (2) alone allows us to account for the zero-resistance phenomenon not making use of equation (1) as London did. Not only is the addition of eq. (1), characterizing perfect conductors, quite superfluous, but even it is in direct contradiction with the nature of superconductivity. Indeed such a nature is totally different from the one of perfect conductivity.

In the latter remark (sect. **5**) we emphasize an equivocation which seems to be present in many works on phenomenological theories of superconductivity [**1**], [**2**], [**11**]. That is the statement that the London theory is a local one.

Although eq. (2) leads to the pointwise relation

$$\boldsymbol{J}(x) = -\frac{\mu}{\Lambda}\,\boldsymbol{A}(x) \qquad x \in \Omega\,, \tag{4}$$

where $\boldsymbol{A}$ is the vector potential for $\boldsymbol{H}$ satisfying the so called « London gauge », this statement is not quite correct since the boundary condition, $\boldsymbol{A}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, in the London gauge already involves a nonlocality. Such a nonlocality is very apparent in the current-magnetic field relationship as we shall see in sect. **5**. Apart from this, we point out that for any given field $\boldsymbol{H}$ on Ω it is necessary to solve the same boundary value problem to obtain either $\boldsymbol{J}$ or $-(\mu/\Lambda)\boldsymbol{A}$, that is

$$\nabla\times\boldsymbol{J} = \nabla\times\Big(-\frac{\mu}{\Lambda}\boldsymbol{A}\Big) = -\frac{\mu}{\Lambda}\boldsymbol{H}\,,\quad \nabla\cdot\boldsymbol{J} = \nabla\cdot\boldsymbol{A} = 0\,,\quad \boldsymbol{J}\cdot\boldsymbol{n} = \boldsymbol{A}\cdot\boldsymbol{n} = 0 \ \text{ on } \partial\Omega\,.$$

Thus eq. (4) seems to be merely an identity.

The aim of the second part of this paper (sects. **6**, **7**) is to formulate a phenomenological theory of superconductivity unifying and in a certain sense simplifying London's and Pippard's ones. To this end we replace equation (2) with the following one

$$\nabla\times(\Lambda\boldsymbol{J}) = -\widetilde{\boldsymbol{B}}\Big(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial}{\partial t}\boldsymbol{J}\Big)\,. \tag{5}$$

When combined with Maxwell's equations eq. (5) formulates the low frequency electromagnetic behaviour of superconductors in such a manner that both London and Pippard formulations can be obtained from it by a suitable choice of the function $\widetilde{\boldsymbol{B}}(\boldsymbol{E}, \boldsymbol{H}, (\partial/\partial t)\boldsymbol{J})$.

Taking into account alternating fields we are able to give the current-magnetic field relationship in both cases by means of suitable dyadic Green's functions. Finally we show that there is no dissipation at low frequencies.

2 - Preliminaries

Let X, Y, Z be material points of a rigid body $\mathscr{B}$ occupying a domain Ω of the three dimensional euclidean space $\mathscr{E}$. We shall identify each material point $X \in \mathscr{B}$ with the geometric point $x = (x_1, x_2, x_3) \in \Omega$ occupied by it. As usual, let $\partial\Omega$ be the boundary of Ω and $\boldsymbol{n}$ be the unit vector normal to $\partial\Omega$. Moreover, let $I = \{t: t \in (0, T)\}$ be the interval spanned by the time variable t.

To describe electromagnetic phenomena inside the body $\mathscr{B}$ we shall introduce the following vector fields defined on $Q = \Omega \times I$: the electric field $\boldsymbol{E}$, the electric displacement $\boldsymbol{D}$, the magnetic field $\boldsymbol{H}$, the magnetic induction $\boldsymbol{B}$, the current density $\boldsymbol{J}$.

Whatever nature the body may have the vector fields listed above must satisfy Faraday's and Ampere's laws. If the fields are smooth enough these laws can be expressed by means of Maxwell's equations

$$(6) \qquad \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}, \qquad\qquad (7) \qquad \nabla \times \boldsymbol{H} = \frac{\partial}{\partial t}\boldsymbol{D} + \boldsymbol{J}.$$

Further relations between the fields are needed to specify electromagnetic features of the material forming the body. Such « constitutive » relations may be stated by functional or differential equations.

Ordinary dielectrics and electromagnetic materials with memory, for example, belong to the former group. The electromagnetic properties of ordinary dielectrics are embodied in the following relations

$$(8) \quad \boldsymbol{D}(x,t) = \tilde{\boldsymbol{D}}(\boldsymbol{E}(x,t), \boldsymbol{H}(x,t)), \qquad (9) \quad \boldsymbol{B}(x,t) = \tilde{\boldsymbol{B}}(\boldsymbol{E}(x,t), \boldsymbol{H}(x,t)).$$

Considering materials with memory functions (8)-(9) have to be replaced by some functionals, that is

$$\boldsymbol{D}(x,t) = \tilde{\boldsymbol{D}}(\boldsymbol{E}^t(x), \boldsymbol{H}^t(x)), \qquad \boldsymbol{B}(x,t) = \tilde{\boldsymbol{B}}(\boldsymbol{E}^t(x), \boldsymbol{H}^t(x)),$$

with $\boldsymbol{E}^t(x)$ and $\boldsymbol{H}^t(x)$ being respectively the « history » of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ up to time t at the point $x \in \Omega$.

On the other hand, we shall see later that some materials such as perfect conductors and superconductors can be specified by some ordinary or partial differential equations. Constitutive relations of such a type belong to the latter groups and have not been much considered.

In the following sections perfect conductors and superconductors are taken into consideration and beyond some physical analogies the difference between their mathematical models is emphasized.

3 - Perfect conductors

The electrodynamics of most isotropic conductors are described by Maxwell's equations along with the following constitutive relations (¹)

$$(10)\quad \boldsymbol{D}(x,t)=\varepsilon(x)\boldsymbol{E}(x,t)\,, \qquad (11)\quad \boldsymbol{B}(x,t)=\mu(x)\boldsymbol{H}(x,t)\,,$$

$$(12)\qquad \lambda\frac{\partial}{\partial t}\boldsymbol{J}(x,t)+r\boldsymbol{J}(x,t)=\boldsymbol{E}(x,t)\,,$$

where ε and μ are respectively the dielectric constant and magnetic permeability of the medium. The constants λ and r are non negative characteristic parameters of the material: r is the normal resistivity, the ratio λ/r is called « relaxation time ». We point out that eq. (12) can be justified on the basis of a microscopic theory (²).

If the electrical resistance is so large that λ is negligible with respect to r, as in normal conductors, then eq. (12) leads to Ohm's law

$$(12)'\qquad \boldsymbol{J}=\sigma\boldsymbol{E}\,,$$

with $\sigma=1/r$ being the normal conductivity.

On the contrary, if the resistivity r is vanishingly small, as in perfect conductors, eq. (12) implies

$$(12)''\qquad \lambda\frac{\partial}{\partial t}\boldsymbol{J}=\boldsymbol{E}\,.$$

Since $\lambda\ll 1$, many results about perfect conductors have been obtained by setting $\boldsymbol{E}=\boldsymbol{0}$ on Ω. Unfortunately, this sharp approximation seems to be insufficient to properly describe conductivity near the boundary of a conducting medium (« skin effect »), and then it will be rejected.

(¹) A nonlinear generalization of them can be given as follows

$$\boldsymbol{D}=\widetilde{\boldsymbol{D}}(\boldsymbol{E},\boldsymbol{H},\boldsymbol{J})\,,\qquad \boldsymbol{B}=\widetilde{\boldsymbol{B}}(\boldsymbol{E},\boldsymbol{H},\boldsymbol{J})\,,\qquad \frac{\partial}{\partial t}\boldsymbol{J}=\boldsymbol{F}(\boldsymbol{E},\boldsymbol{H},\boldsymbol{J})\,.$$

(²) See for example [**18**], pp. 97-98.

Assuming $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{J}$ as independent variables and using relations (10), (11), the system of partial differential equations (6), (7), (12)″ may be solved. We shall now show that the corresponding initial-boundary value problem with initial conditions

$$(13)\qquad \boldsymbol{E}(x,0)=\boldsymbol{E}_0(x)\,,\quad \boldsymbol{H}(x,0)=\boldsymbol{H}_0(x)\,,\quad \boldsymbol{J}(x,0)=\boldsymbol{J}_0(x)\quad \text{on } \Omega,$$

and boundary condition

$$(14)\qquad \boldsymbol{E}\times\boldsymbol{n}=\boldsymbol{0}\quad \text{on } \partial\Omega\times I,$$

is well posed.

We first note that by means of (10), (11) and (12)″ Poynting's theorem, stated by

$$(15)\qquad -\int_{\partial\Omega}\boldsymbol{E}\times\boldsymbol{H}\cdot\boldsymbol{n}\,\mathrm{d}\sigma=\int_{\Omega}\Big(\frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{H}+\frac{\partial\boldsymbol{D}}{\partial t}\cdot\boldsymbol{E}+\boldsymbol{J}\cdot\boldsymbol{E}\Big)\,\mathrm{d}x\,,$$

leads to the following balance equation

$$(16)\qquad -\int_{\partial\Omega}\boldsymbol{E}\times\boldsymbol{H}\cdot\boldsymbol{n}\,\mathrm{d}\sigma=\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\tfrac{1}{2}\int_{\Omega}(\mu H^2+\varepsilon E^2+\lambda J^2)\,\mathrm{d}x\Big\}\,.$$

Thus the electromagnetic energy defined by $\frac{1}{2}\int_{\Omega}(\mu H^2+\varepsilon E^2+\lambda J^2)\,\mathrm{d}x$ is conserved if (14) holds on the boundary. The lack of dissipation proves that eq. (12)″ successfully explains the experimental behaviour of perfect conductors. Integrating (15) over I and using relations (13), (14) we get

$$(17)\qquad \int_{\Omega}(\mu H^2(T)+\varepsilon E^2(T)+\lambda J^2(T))\,\mathrm{d}x=\int_{\Omega}(\mu H_0^2+\varepsilon E_0^2+\lambda J_0^2)\,\mathrm{d}x\,.$$

With homogeneous initial conditions, that is $\boldsymbol{E}_0=\boldsymbol{H}_0=\boldsymbol{J}_0\equiv\boldsymbol{0}$ on Ω, the relation (17) leads to the vanishing of $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}$ almost everywhere on $Q=\Omega\times I$.

We thus see that the initial-boundary value problem stated below uniquely determines the currents and the fields in a perfect conductor. Because of the linearity of such a system it does not seem difficult to prove an existence theorem as well.

4 - Superconductors

Two-fluid models have played a prominent role in the development of our understanding of both quantum liquids: the electron fluid in superconductors and liquid helium. In the case of superconductors the electrodynamics of a two-fluid model was worked out by F. and H. London [8]$_{1,2}$. The London theory assumes that the current in a superconductor, $\boldsymbol{J}$, is composed of a « supercurrent » $\boldsymbol{J}_s$ and a normal current $\boldsymbol{J}_n$. The former is due to the « superconducting » portion of the conduction electrons and is ruled by London equations (1), (2) while the latter obeys Ohm's law (12)′.

As a simplifying assumption we shall ignore the contribution due to $\boldsymbol{J}_n$ so that $\boldsymbol{J} = \boldsymbol{J}_s$ (3). Nevertheless, an exhaustive treatment would involve no serious difficulties. Moreover we shall simply assume that there is total charge neutrality, leading to the vanishing of ϱ and $\nabla \cdot \boldsymbol{J}$. This is eminently reasonable and usually accepted [1], [16].

Under these assumptions, in order to develop an electrodynamics of superconductors on a phenomenological basis, we suggest the following constitutive equations

$$\nabla \times (\Lambda \boldsymbol{J}) = -\boldsymbol{B}\,, \tag{18}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}\,, \tag{19}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} + \nu \frac{\partial}{\partial t} \boldsymbol{J}\,, \tag{20}$$

with ν being a non negative constitutive parameter depending on impurity contenent (4). Furthermore ν is supposed to be vanishingly small in pure materials. In this case eq. (20) reduces to

$$\boldsymbol{B} = \mu \boldsymbol{H}\,, \tag{21}$$

so currents and fields in pure specimens are related by London's equation (2).

Since the value of Λ is about 10^{-10} abH·cm (see [4], p. 28) we note that the vanishing of the field $\boldsymbol{B}$ deep inside a macroscopic superconductor is always contained in eq. (18).

We shall show now that for the quasi-static case eqs. (18), (19), (20) are able to account for both Meissner effect and zero-resistance phenomenon.

Because of static conditions all time derivatives vanish, so that by combining eqs. (18), (20) we get eq. (2). It turns out that the second London equation covers not only the Meissner effect in bulk superconductors, but also the magnetic properties of samples with dimensions comparable to the « pene-

(3) This is really the case in static conditions and in alternating fields at low frequencies (less than 10^{10} Hz). See also [16], p. 247 and [4], p. 18.

(4) This means that ν varies with electronic mean free path, unlike ε, μ and Λ.

tration depth » λ (see [1], [4], [9]). Comparison with the London theory [17] leads to the relation

(22) $$\lambda = \sqrt{\frac{\Lambda}{\mu}} \ (\sim 10^{-6}\ \text{cm}) .$$

To account for the zero-resistance phenomenon the London theory retains eq. (1), borrowing it from the «perfect conductors» theory (see eq. (12)″). As emphasized by London eq. (1) does not always follow from eq. (2) and must be considered as an independent postulate. Nevertheless eq. (2) it self implies that no dissipation occurs in superconductors. The proof is as follows.

Differentiating eq. (18) with respect to time and using Maxwell's equation (6), we have

(23) $$\nabla \times \frac{\partial}{\partial t}(\Lambda \boldsymbol{J}) = \nabla \times \boldsymbol{E} .$$

By solving eq. (23) for a simply connected superconductor in a domain $\Omega \subset R^3$ t can be shown that

(24) $$\frac{\partial}{\partial t}(\Lambda \boldsymbol{J}) = \boldsymbol{E} + \nabla \Phi ,$$

with Φ being any smooth scalar function defined on $Q = \Omega \times I$. Then, making use of eqs. (19), (20) and substituting for $\boldsymbol{E}$ from eq. (24), Poynting's theorem (see (15)) leads to

(25) $$-\int_{\partial\Omega} \boldsymbol{E} \times \boldsymbol{H} \cdot \boldsymbol{n}\, d\sigma = \frac{d}{dt}\{\tfrac{1}{2}\int_{\Omega}(\mu H^2 + \varepsilon E^2 + \Lambda J^2)\, dx\} + \int_{\Omega} \nu \boldsymbol{H} \cdot \frac{\partial^2}{\partial t^2} \boldsymbol{J}\, dx - \int_{\Omega} \boldsymbol{J} \cdot \nabla \Phi\, dx .$$

Assuming as usual

(26) $$\boldsymbol{J} \cdot \boldsymbol{n} = 0 ,$$

on the boundary $\partial\Omega$ of an insulated superconductor, since

(27) $$\nabla \cdot \boldsymbol{J} = 0 \quad \text{on } \Omega ,$$

the last integral on the right hand side vanishes

$$\int_{\Omega} \boldsymbol{J} \cdot \nabla \Phi\, dx = \int_{\partial\Omega} \Phi \boldsymbol{J} \cdot \boldsymbol{n}\, d\sigma - \int_{\Omega} \Phi \nabla \cdot \boldsymbol{J}\, dx = 0 .$$

Thus there is no Joule dissipation.

Supercurrents preservation therefore follows in the stationary case as well as in pure samples. As we shall see later (sect. **6**), the same result approximately holds even in alternating field at low frequencies.

Not only the zero-resistance phenomenon but even the absence of any « Hall effect » can be accounted for by eqs. (18), (19), (20) not resorting to perfect conductivity arguments [**1**].

Because of the charge neutrality, $\varrho = 0$, the electromagnetic momentum balance leads to

$$\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}) + \boldsymbol{J}\times\boldsymbol{B} = \nabla\cdot\boldsymbol{T}, \tag{28}$$

where $\boldsymbol{T}$ is the Maxwell stress tensor, defined by

$$\boldsymbol{T} = \boldsymbol{E}\otimes\boldsymbol{D} + \boldsymbol{H}\otimes\boldsymbol{B} - \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})\boldsymbol{I}.$$

From eqs. (18), (27) it follows that

$$\boldsymbol{J}\times\boldsymbol{B} = -\Lambda\boldsymbol{J}\times(\nabla\times\boldsymbol{J}) = -\nabla\cdot\boldsymbol{S}, \tag{29}$$

with $\boldsymbol{S} = \Lambda(\frac{1}{2}J^2\boldsymbol{I} - \boldsymbol{J}\otimes\boldsymbol{J})$ being the so called « London stress tensor ». Equation (28) may therefore be rewritten as

$$\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}) = \nabla\cdot(\boldsymbol{T}+\boldsymbol{S}) \quad \text{and under stationary conditions} \quad \nabla\cdot(\boldsymbol{T}+\boldsymbol{S}) = 0\,.$$

Thus there are no volume forces acting on the superconductor. Further, eq. (29) also implies that the Lorentz force is exactly balanced by the inertial force $-\Lambda\boldsymbol{J}\times(\nabla\times\boldsymbol{J})$, so that there will be no Hall effect in a superconductor.

5 - Nonlocality of the London theory

As F. London showed [**7**], the second London equation takes a particularly simple form when one introduces a vector potential $\boldsymbol{A}$ for the magnetic field $\boldsymbol{H}$

$$\nabla\times\boldsymbol{A} = \boldsymbol{H}. \tag{30}$$

Replacing the field by $\boldsymbol{A}$ and choosing a gauge such that

$$\nabla\cdot\boldsymbol{A} = 0\,, \tag{31}$$ $$\boldsymbol{A}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,, \tag{32}$$

equation (2) reduces to

$$J = -\frac{\mu}{\Lambda} A . \tag{33}$$

The same result leads even from eq. (18) by introducing a vector potential A for $1/\mu B$

$$\nabla \times A = \frac{1}{\mu} B . \tag{30$'$}$$

The gauge (31), (32) is sometimes referred to as the «London gauge».

Applying the pointwise relation (33) to eqs. (18), (19), (21) it turns out that the values of D, B, H at $x \in \Omega$ can be obtained by specifying just the values of E, H, A at the same point. Inspite of this, it is not quite correct, as has sometimes be done, to call eq. (33) a «local relation». Indeed, the London gange with boundary condition (32) implies that A is specified at more than just the point at which J is being measured, and hence already involves a nonlocality [4].

Apart from these reasons, we note that the same boundary value problem has to be solved in order to obtain J as well as $-(\mu/\Lambda)A$ if H (or B) is known. Indeed, given H on Ω, the linear system (2), (27) (or (18), (27) if B is given) with boundary condition (26) is identical to (30), (31), (32) (or (30)$'$, (31), (32)) which defines the vector potential. Hence, the statement (33) is merely an identity and thus can not represent any constitutive relation.

It will be shown here that the nonlocal nature of the second London equation is emphasized by the current-field relation solving eqs. (2), (27) for a bulk superconductor.

By introducing the parameter λ defined in (22) we get

$$\nabla \times J = -\frac{1}{\lambda^2} H , \qquad \nabla \cdot J = 0 . \tag{34}$$

Moreover, the following boundary condition

$$J = \frac{1}{\lambda} n \times H \quad \text{on } \partial\Omega , \tag{35}$$

is assumed to be satisfied on the surface of an insulated superconductor. It turns out that such a condition implies (26) and is a very close approximation when the radius of curvature of the surface is large compared with the penetration depth λ (5).

It can be shown that for any given solenoidal field $\boldsymbol{H}$ on $\bar{\Omega} = \Omega \cup \partial\Omega$ there is one and only one solution $\boldsymbol{J}$ to the problem (34), (35).

In order to prove uniqueness let us consider the difference $\boldsymbol{J} = \boldsymbol{J}_1 - \boldsymbol{J}_2$ between solutions $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ of (34), (35). Such a function satisfies the homogeneous problem

$$\nabla \times \boldsymbol{J} = 0 \,, \quad \nabla \cdot \boldsymbol{J} = 0 \,, \quad \boldsymbol{J} = 0 \quad \text{on } \partial\Omega \,,$$

which leads to $\boldsymbol{J} = 0$ in Ω, as it is well known.

Now we want to express the solution $\boldsymbol{J}$ at a point $x' \in \Omega$ as a function of the whole field $\boldsymbol{H}$ on Ω. To this end we note that $(34)_2$ implies the existence of a sufficiently smooth field $\boldsymbol{K}$ such that

$$\nabla \times \boldsymbol{K} = \boldsymbol{J} \,. \tag{36}$$

It follows from (34), (35) that

$$\nabla \times \nabla \times \boldsymbol{K} = -\frac{1}{\lambda^2}\boldsymbol{H} \quad \text{in } \Omega \,, \tag{37}$$

$$(\nabla \times \boldsymbol{K}) = \frac{1}{\lambda}\boldsymbol{n} \times \boldsymbol{H} \quad \text{on } \partial\Omega \,. \tag{38}$$

Let us consider the identity

$$\int_{\partial\Omega} [\boldsymbol{n} \times (\nabla \times \boldsymbol{K}) \cdot \boldsymbol{\Gamma} + \boldsymbol{n} \times \boldsymbol{K} \cdot (\nabla \times \boldsymbol{\Gamma})] \, d\sigma = \int_{\Omega} [\nabla \times \nabla \times \boldsymbol{K} \cdot \boldsymbol{\Gamma} - \boldsymbol{K} \cdot \nabla \times \nabla \times \boldsymbol{\Gamma}] \, dv \,, \tag{39}$$

where $\boldsymbol{\Gamma}(x, x')$ is a tensor defined on $R^3 \times R^3$. If we choose

$$\boldsymbol{\Gamma} = G_0 \boldsymbol{I} \,, \tag{40}$$

with $\boldsymbol{I}$ being the unit tensor and $G_0 = 1/4\pi R$, $R = |x - x'|$, being the solution of the problem

$$\Delta G_0(R) = -\delta(R) \,, \quad \lim_{R \to \infty} G_0(R) = 0 \,, \tag{41}$$

(5) See for example [1], pp. 290-293, [4], pp. 29-31.

then (39) leads to

$$K(x') = -\frac{1}{\lambda^2}\int_\Omega G_0 H\,dv + \frac{1}{\lambda}\int_{\partial\Omega} G_0 H_t\,d\sigma - \int_v K\cdot\nabla\nabla G_0\,dv$$

$$-\int_{\partial\Omega}(n\times K)\cdot\nabla\times G_0 I\,d\sigma \qquad x'\in\Omega\,,$$

where $H_t = (n\times H)\times n$ is the component of H parallel to $\partial\Omega$. By means of (36) we get $J(x') = \nabla'\times K(x')$ $(\nabla'\equiv\partial/\partial x')$, thus it follows that

$$J(x') = -\frac{1}{\lambda^2}\int_\Omega \nabla' G_0\times H\,dv + \frac{1}{\lambda}\int_{\partial\Omega}\nabla' G_0\times H_t\,d\sigma - \int_\Omega K\cdot\nabla'\times\nabla\nabla G_0\,dv$$
$$-\int_{\partial\Omega}(n\times K)\cdot\nabla'\times\nabla\times G_0 I\,d\sigma\,. \tag{42}$$

A straightforward computation yields the vanishing of the last two integrals on the right-hand. Indeed, using (41) the last integral reduces to

$$-\int_{\partial\Omega}(n\times K)\cdot\nabla'\nabla G_0\,d\sigma = -\int_{\partial\Omega}(n\cdot\nabla\times K)\nabla' G_0\,d\sigma$$

and then (38) allows it to vanish. As a result we arrive at the following current-field relation

$$J(x') = -\frac{1}{\lambda^2}\int_\Omega \nabla' G_0\times H\,dv + \frac{1}{\lambda}\int_{\partial\Omega}\nabla' G_0\times H_t\,d\sigma, \tag{43}$$

at any point $x'\in\Omega$.

6 - General constitutive equations

Let us consider now the general constitutive equations (18), (19), (20) we have previously proposed for superconducting materials.

We shall deal with the following boundary value problem arising when fields depend on the time by a common complex factor $\exp[+i\omega t]$

$$\nabla\times J + i\gamma J = -\frac{1}{\lambda^2}H,\qquad \nabla\cdot J = 0\,, \tag{44}$$

$$J = \frac{1}{\lambda}\,n\times H\qquad \text{on } \partial\Omega\,, \tag{45}$$

where $\lambda^2 = \Lambda/\mu$ and $\gamma = \nu\omega/\Lambda$. In such a problem $\boldsymbol{H}$ and $\boldsymbol{J}$ are complex vector valued functions depending on x alone. Henceforth $\boldsymbol{H}$ is supposed to be a solenoidal field in order that $\nabla\cdot\boldsymbol{J} = 0$ may be obtained as a consequence of eq. (44).

Under these assumption and for any given solenoidal field $\boldsymbol{H}$ we want to solve the problem (44)-(45) expressing its solutions by means of suitable dyadic Green's functions.

First, in order to prove uniqueness of solutions, we take into consideration the problem

(46) $\nabla\times\boldsymbol{J} + i\gamma\boldsymbol{J} = 0\,,$ (47) $\boldsymbol{J} = 0 \quad \text{on } \partial\Omega\,,$

where $\boldsymbol{J} = \boldsymbol{J}_1 - \boldsymbol{J}_2$ is the difference between two solutions of (44), (45) corresponding to the same choice of $\boldsymbol{H}$.

Taking the curl of (46) and multiplying the result by $\boldsymbol{J}^*$, the conjugate of $\boldsymbol{J}$, then integrating over Ω leads to

$$\begin{aligned} 0 &= \int_\Omega (\nabla\times\nabla\times\boldsymbol{J} + i\gamma\nabla\times\boldsymbol{J})\cdot\boldsymbol{J}^*\,\mathrm{d}v \\ &= \int_{\partial\Omega} \boldsymbol{n}\cdot(\nabla\times\boldsymbol{J})\times\boldsymbol{J}^*\,\mathrm{d}\sigma + \int_\Omega (\nabla\times\boldsymbol{J}\cdot\nabla\times\boldsymbol{J}^* + \gamma^2\boldsymbol{J}\cdot\boldsymbol{J}^*)\,\mathrm{d}v\,. \end{aligned}$$

Using (47) it follows that,

$$\int_\Omega |\nabla\times\boldsymbol{J}|^2\,\mathrm{d}v + \gamma^2\int_\Omega |\boldsymbol{J}|^2\,\mathrm{d}v = 0\,,$$

and then $\boldsymbol{J}$ must vanish.

On the other hand, in order to express $\boldsymbol{J}(x)$ as a function of $\boldsymbol{H}$ we shall introduce a dyadic Green's function $\boldsymbol{\Gamma}(x, x')$ satisfying

$$\nabla\times\boldsymbol{\Gamma} + i\gamma\boldsymbol{\Gamma} = -\delta\boldsymbol{I} \quad \text{in } \mathbb{R}^3\,, \tag{48}$$

for all $x'\in\mathbb{R}^3$, where $\delta = \delta(x - x')$ is the Dirac delta function and $\boldsymbol{I}$ the unit tensor. Further we suppose

$$\lim_{|x-x'|\to\infty} \boldsymbol{\Gamma}(x, x') = 0\,. \tag{49}$$

Taking $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_1 + i\boldsymbol{\Gamma}_2$, with $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ being real tensors, we get

$$\nabla\times\boldsymbol{\Gamma}_1 - \gamma\boldsymbol{\Gamma}_2 = -\delta\boldsymbol{I}\,, \qquad \nabla\times\boldsymbol{\Gamma}_2 + \gamma\boldsymbol{\Gamma}_1 = 0\,. \tag{50}$$

Substituting for Γ_1 from $(50)_2$ into $(50)_1$ yields

$$\nabla\times\nabla\boldsymbol{\Gamma}_2 + \gamma^2\boldsymbol{\Gamma}_2 = \gamma\delta\boldsymbol{I}\,. \tag{51}$$

It turns out that a solution of eq. (51) satisfying (49) can be expressed as follows ([6])

$$\Gamma_2 = \gamma\,[\boldsymbol{I} + \frac{1}{\gamma^2}\nabla\nabla']\,G\,, \tag{52}$$

where $G = (1/4\pi R)\exp[-\gamma R]$, $R = |x - x'|$. Essentially, this result rests on the fact that G solves $(\Delta - \gamma^2)G = -\delta$ in $\mathbb{R}^3$. As a consequence we get

$$\Gamma_1 = -\nabla \times (G\boldsymbol{I})\,. \tag{53}$$

Substituting eqs. (44)-(48) into the identity

$$\int_{\partial\Omega} \boldsymbol{n}\cdot(\boldsymbol{J}\times\boldsymbol{\Gamma})\,d\sigma = \int_{\Omega}(\nabla\times\boldsymbol{J}\cdot\boldsymbol{\Gamma} - \boldsymbol{J}\cdot\nabla\times\boldsymbol{\Gamma})\,dv\,,$$

a straightforward computation yields the following relation

$$\boldsymbol{J}(x') = \frac{1}{\lambda^2}\int_{\Omega}\boldsymbol{H}(x)\cdot\boldsymbol{\Gamma}(x, x')\,dv - \int_{\partial\Omega}[\boldsymbol{n}(x)\times\boldsymbol{J}(x)]\cdot\boldsymbol{\Gamma}(x, x')\,d\sigma\,,$$

at any point $x' \in \Omega$. Hence, by means of (45), it follows that

$$\boldsymbol{J}(x') = \frac{1}{\lambda^2}\int_{\Omega}\boldsymbol{H}(x)\cdot\boldsymbol{\Gamma}(x, x')\,dv - \frac{1}{\lambda}\int_{\partial\Omega}\boldsymbol{H}_t(x)\cdot\boldsymbol{\Gamma}(x, x')\,d\sigma\,, \tag{54}$$

where $\boldsymbol{H}_t = (\boldsymbol{n}\times\boldsymbol{H})\times\boldsymbol{n}$.

Assuming that $\Omega \equiv \mathbb{R}^3$ and introducing a vector potential $\boldsymbol{A}$ such that $\nabla\times\boldsymbol{A} = \boldsymbol{H}$ and $\nabla\cdot\boldsymbol{A} = 0$, then by means of (49) from eq. (54) we obtain

$$\boldsymbol{J}(x') = \frac{1}{\lambda^2}\int_{\Omega}\boldsymbol{A}(x)\cdot\nabla\times\boldsymbol{\Gamma}(x, x')\,dv\,.$$

Since $\nabla\times\boldsymbol{\Gamma} = -\nabla\nabla G + \Delta G\boldsymbol{I} + i\gamma\nabla G\times\boldsymbol{I}$ and $G = (1/4\pi R)\exp[-\gamma R]$, setting $\boldsymbol{F}(R)\exp[-\gamma R] = -\nabla\nabla G + \Delta G\boldsymbol{I} + i\gamma\nabla G\times\boldsymbol{I}$ we have the following relation

$$\boldsymbol{J}(x') = \frac{1}{\lambda^2}\int_{\Omega}\boldsymbol{A}\cdot\boldsymbol{F}\exp[-\gamma R]\,dv\,, \tag{55}$$

which is very similar to (3) if we assume $\xi = 1/\gamma$.

([6]) See for example [**6**], Appendix I, pp. 385-386.

Our aim is to show now that a current-field relation such as (54) implies that no energy dissipation occurs at low frequencies. Taking into account harmonic fields, Poynting's theorem (25) may be rewritten as follows

$$\int_{\partial\Omega} \boldsymbol{n}\cdot\boldsymbol{E}\times\boldsymbol{H}^* \,d\sigma = i\omega\int_{\Omega}(\varepsilon|\boldsymbol{E}|^2 - \mu|\boldsymbol{H}|^2 - \Lambda|\boldsymbol{J}|^2 - i\omega\nu\boldsymbol{J}\cdot\boldsymbol{H}^*)\,dv\,.$$

In order to express the term $i\omega\nu\boldsymbol{J}\cdot\boldsymbol{H}^*$ in a suitable form, we give the expression of $\boldsymbol{\Gamma}$ as a function of γ; that is

$$\boldsymbol{\Gamma}(\gamma) = \frac{1}{\gamma}\exp[-\gamma R](\boldsymbol{T}_0 + \gamma\boldsymbol{T}_1 + \gamma^2\boldsymbol{T}_2)\,, \tag{56}$$

where

$$\boldsymbol{T}_0 = -i\nabla\nabla' G_0\,,\qquad \boldsymbol{T}_1 = \nabla G_0\times\boldsymbol{I} + i(G_0\nabla\nabla' R + 2\nabla G_0\otimes\nabla' R)\,,$$

$$\boldsymbol{T}_2 = -G_0[\nabla R\times\boldsymbol{I} + i(\nabla' R\otimes\nabla R + \boldsymbol{I})$$

are tensors independent of γ since $G_0 = 1/4\pi R$. By means of (56) we obtain

$$i\omega\nu\boldsymbol{J} = i\frac{\omega\nu}{\lambda^2}\Big[\int_{\Omega}\boldsymbol{H}\cdot\boldsymbol{\Gamma}\,dv - \lambda\int_{\partial\Omega}\boldsymbol{H}_t\cdot\boldsymbol{\Gamma}\,d\sigma g\Big] = i\frac{\omega\nu}{\gamma\lambda^2}[\boldsymbol{I}_0 + \gamma\boldsymbol{I}_1 + \gamma^2\boldsymbol{I}_2]\,,$$

where $\boldsymbol{I}_\alpha = \int_{\Omega}\boldsymbol{H}\cdot\boldsymbol{T}_\alpha\exp[-\gamma R]\,dv - \lambda\int_{\partial\Omega}\boldsymbol{H}_t\cdot\boldsymbol{T}_\alpha\exp[-\gamma R]\,d\sigma,$ for $\alpha = 0, 1, 2\,.$

Setting now $\xi = 1/\gamma$ and remembering that $\lambda^2 = \Lambda/\mu$ and $\gamma = \omega\nu/\Lambda$, it turns out that

$$i\omega\nu\boldsymbol{J} = i\mu\Big[\boldsymbol{I}_0 + \frac{1}{\xi}\boldsymbol{I}_1 + \frac{1}{\xi^2}\boldsymbol{I}_2\Big]\,. \tag{57}$$

If we choose the parameter ν such that $\nu/\Lambda \approx 1/c$, with $c = 1/\sqrt{\varepsilon\mu}$, then from the assumption (7) $\omega \ll c/\lambda$ it easily follows that the «coherence length» ξ is much larger than the «penetration depth» λ ($\xi \gg \lambda$). Hence, we can ignore the terms $1/\xi\boldsymbol{I}_1$ and $1/\xi^2\boldsymbol{I}_2$ of (57) and the surface integral in $\boldsymbol{I}_0$ so that

$$i\omega\nu\boldsymbol{J} \approx \mu\int_{\Omega}\boldsymbol{H}\cdot\nabla\nabla' G_0\exp[-R/\xi]\,dv\,.$$

In order to complete our proof we point out that the quantity

$$i\omega\nu\int_{\Omega}\boldsymbol{J}\cdot\boldsymbol{H}^*\,dv' \approx \mu\int_{\Omega}\int_{\Omega}\boldsymbol{H}\cdot\nabla\nabla' G_0\exp[-R/\xi]\boldsymbol{H}^*\,dv\,dv'$$

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(7) It is interesting to note that in the case of a plane boundary such an assumption leads to a penetration depth which is approximately equal to λ.

is a real number. Indeed, since the kernel $K(x, x') = \nabla\nabla' G_0(R) \exp[-R/\xi]$ is a real symmetric tensor such that $K(x, x') = K(x', x)$, it turns out that

$$\big(H(x)\cdot K(x, x')\cdot H^*(x')\big)^* = H(x')\cdot K(x', x)\cdot H^*(x) \, .$$

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