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On the differential of the Kähler form (**)

A LUIGI CAPRIOLI per il suo 70° compleanno

1 - Introduction

Let V be an almost hermitian manifold, J the almost complex structure, g the riemannian structure. We denote by DJ the tensor field of type $(1, 2)$, deduced from J by covariant differentiation in the riemannian connection.

Let ω be the Kähler form. We denote by K the tensor field of type $(1, 2)$ corresponding to $d\omega$ (*Kähler field*).

The main point of the paper is to show that some known classes of almost hermitian manifolds can be characterized, by assigning particular expressions to the Kähler field K (Sec. **6**; Th. 4, Th. 5, Th. 6, Th. 7).

Some preliminary results are obtained in Sec. **4**. More explicitly, we show that the combined action on DJ of the isomorphisms α, W, λ, γ, introduced in previous papers, leads to characterize some of the mentioned classes of manifolds (Th. 1, Th. 2, Th. 3). These characterizations however do not involve the Kähler field K.

Finally, lemmas L_1, L_2 and theorems Th. 2, Th. 3, Th. 6 show that there exists a sort of symmetry between Hermite manifolds and almost Kōto manifolds (quasi-Kähler manifolds).

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2 - Isomorphisms α, W, λ, γ

Let V be an *almost Hermite manifold* of dimension $2n$ and class C^{2n+1} (1). Let $\mathscr{T}^r_s$ be the linear space of the tensor fields of type (r, s) on V. In particular, let g be the symmetric field of $\mathscr{T}^0_2$ of class C^1, defining the riemannian metric on V, and let J be the tensor field of $\mathscr{T}^1_1$ of class C^{2n}, defining the almost complex structure on V.

We make use in the following of the *isomorphisms* α, W, λ, γ of the linear space $\mathscr{T}^1_2$, introduced in previous papers (2). The first one depends on the differential structure of V; W, λ depend on the almost complex structure and γ on the riemannian structure of V.

We denote then by $\mathrm{D}J$ the tensor field of $\mathscr{T}^1_2$, obtained from J by covariant differentiation with respect to the Levi-Civita connection (3).

3 - Kähler tensor field

Let ω be the *Kähler form* and K the *Kähler tensor field*, i.e. the tensor field of $\mathscr{T}^1_2$ corresponding to $\mathrm{d}\omega$.

More explicitly

$$\omega = c^1_1(g \otimes J)\,, \tag{1}$$

and

$$K = c^1_3(G \otimes \mathrm{d}\omega)\,, \tag{2}$$

where c^r_s denotes contraction, G is the symmetric tensor field of $\mathscr{T}^2_0$ defined by $c^1_1(G \otimes g) = \delta$ and δ is the classical Kronecker field.

The isomorphisms α, γ of Sec. **2** permit us to express the tensor field K in terms of $\mathrm{D}J$

$$K = (1 - \alpha - \gamma)\mathrm{D}J \text{ (4)}\,. \tag{3}$$

Another expression of K is the following

$$K = \mathrm{D}J + (\alpha\gamma + \gamma\alpha)\mathrm{D}J \text{ (5)}\,. \tag{4}$$

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(1) For the basic facts about almost Hermite manifolds see K. Yano [**6**], ch. 9; S. Kobayashi and K. Nomizu [**3**], II, ch. 9.

(2) For the definitions and the essential relations concerning α, W, λ, γ, see G. B. Rizza [**5**]$_4$, Sec. **2**.

(3) Relations and properties involving $\mathrm{D}J$ can be found in G. B. Rizza [**5**]$_4$, Sec. **3**.

(4) See for ex. G. B. Rizza [**5**]$_4$ (12), Sec. **3**.

(5) To prove that (3) and (4) are equivalent, just use relations (8), $(1)_1$ of [**5**]$_4$ and the skew-symmetry of $\gamma\mathrm{D}J$ ([**5**]$_3$, p. 873).

Finally, it is worth recalling the *relations*

(5) $$\alpha K = -K\,, \quad \gamma K = -K\ (^6)\,.$$

4 - First results

Many classes of almost Hermite manifolds, generalizing the class of Kähler manifolds, are known in the literature.

Some of them occur in this paper. More explicity, we consider G_1 *spaces* (*under-Kähler manifolds*), G_2*-spaces*, *Hermite manifolds*, *almost Kōto manifolds* (*quasi-Kähler manifolds*), *almost Tachibana manifolds* (*nearly Kähler manifolds*) *and almost Kähler manifolds* (7).

All the above classes appear in the classification due to A. Gray and L. M. Hervella (8) and have been characterized and studied by various Authors, using different techniques (9).

We add here three more *characterization theorems*, that will be useful in the sequel.

First of all, we introduce the algebra $\mathscr{H}$ of the homomorphisms of $\mathscr{T}^1_2$ in itself and consider in $\mathscr{H}$ the *symmetric product* and the *Lie product*, defined by

$$(A, B) = AB + BA\,, \quad [A, B] = AB - BA\,,$$

where A, B are arbitrary elements of $\mathscr{H}$.

Then, we state the lemmas

L_1. *In an almost Hermite manifold, conditions*

a$_1$ $$(\alpha, W)DJ = 0\,,$$

a$_2$ $$(W, \gamma)DJ = 0\,,$$

a$_3$ $$(\lambda, \gamma)DJ = 0$$

are equivalent.

(6) The first one expresses the fact that K is a skew-symmetric field ([**5**]$_3$, p. 873). For the second relation see [**5**]$_3$, (9)$_3$, p. 872.

(7) For the definitions, see G. B. Rizza [**5**]$_4$, Sec. 5. Hermite manifolds can be defined by condition $(1 - \alpha W\alpha)DJ = 0$; see G. B. Rizza [**5**]$_2$, p. 473.

(8) A. Gray and L. M. Hervella [**2**].

(9) For example, a particular class of connections occurs in [**5**]$_1$ and in [**4**], convenient almost complex conditions are used in [**5**]$_{2,3}$, while the Nijenhuis tensor field N plays an essential role in [**5**]$_4$.

L_2. *In an almost Hermite manifold, conditions*

$\mathbf{b}_1$ $$[\alpha, W]DJ = 0\ ,$$

$\mathbf{b}_2$ $$[W, \gamma]DJ = 0\ ,$$

$\mathbf{b}_3$ $$[\lambda, \gamma]DJ \ = 0$$

are equivalent.

Now, we are able to state the *theorems*

Th. 1. *If V is an almost Tachibana manifold, then*

$$DJ = -\gamma DJ \tag{6}$$

and conversely.

Th. 2. *If V is a Hermite manifold, then the field* DJ *satisfies condition* $\mathbf{a}_r$ $(r = 1, 2, 3)$; *and conversely.*

Th. 3. *If V is an almost Kōto manifold, then the field* DJ *satisfies condition* $\mathbf{b}_r$ $(r = 1, 2, 3)$; *and conversely.*

It is worth remarking that from L_1, L_2, Th. 2, Th. 3 it appears a sort of analogy between Hermite manifolds and almost Kōto manifolds (10).

5 - Proofs

The first step to prove lemma L_1 is to show that, respectively, conditions $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are *equivalent* to conditions

$\mathbf{a}'_1$ $$(\alpha W \alpha W + 1)DJ = 0\ ,$$

$\mathbf{a}'_2$ $$(W\gamma W\gamma + 1)DJ = 0\ ,$$

$\mathbf{a}'_3$ $$(\gamma\lambda\gamma\lambda - 1)DJ \ = 0\ .$$

The action of the isomorphisms $aW, W\gamma, \gamma\lambda$ on $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and the use of relation (1) of **[5]**$_4$ prove that $\mathbf{a}_r$ implies $\mathbf{a}'_r$ $(r = 1, 2, 3)$; The action of the isomorphisms $W\alpha, \gamma W, \lambda\gamma$ on $\mathbf{a}'_1$, $\mathbf{a}'_2$, $\mathbf{a}'_3$ and again the use of relation (1) of **[5]**$_4$ prove the converse.

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(10) This analogy has been noticed since 1977 (G. B. Rizza **[5]**$_2$, Sec. **6**, p. 274).

The second step is to show that each condition $\mathbf{a}'_r$ (r = 1, 2, 3) is *equivalent* to condition

$$\mathbf{a} \qquad (1-\alpha W\alpha)\mathrm{D}J=0\ .$$

The fact is an easy consequence of relations (1), (3), (4) of $[\mathbf{5}]_4$ and of relation (6) of the same paper.

The proof of lemma L_2 is similar. Each condition $\mathbf{b}_r$ (r = 1, 2, 3) results to be *equivalent* to condition

$$\mathbf{b} \qquad (1+\alpha W\alpha)\mathrm{D}J=0\ .$$

Now, Th. 2 and Th. 3 follow immediately from the proof of lemmas L_2, L_3. It is sufficient to remember that conditions **a** and **b** define Hermite manifolds and almost Kōto manifolds, respectively (11).

Finally, we prove Th. 1. If V is an almost Tachibana manifold, taking into account relation (8) of $[\mathbf{5}]_4$ and the skew-symmetry of the fields $\mathrm{D}J$, $\gamma\mathrm{D}J$, we have (12)

$$\mathrm{D}J=-\alpha\gamma\alpha\mathrm{D}J=\alpha\gamma\mathrm{D}J=-\gamma\mathrm{D}J\ .$$

Conversely, since $\gamma\mathrm{D}J$ is a skew-symmetric field, from relation (6) we get

$$\sigma\mathrm{D}J=-\sigma\gamma\mathrm{D}J=0\ ,$$

so V is an almost Tachibana manifold.

6 - Characterization theorems

We are able now to show that the classes of almost Hermite manifolds mentioned in Sec. 4 can be characterized by particular expressions for the Kähler field K.

For the sake of completeness, we recall first that *if* $K=0$, *then* V *is an almost Kähler manifold; and conversely* (13).

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(11) See G. B. Rizza $[\mathbf{5}]_2$, Sec. **4**.

(12) See G. B. Rizza $[\mathbf{5}]_4$, $(20)_1$, where $2\sigma=1+\alpha$. See also G. B. Rizza $[\mathbf{5}]_3$, p. 873.

(13) See A. Gray and L. M. Hervella [**2**], p. 40-41, where the Kähler form has been denoted by F.

Next, we list the *theorems*

Th. 4. *If the Kähler field has the form*

(7) $$K = -\gamma \mathrm{D}J + 2(\alpha W + W\alpha)\mathrm{D}J + (\alpha W\alpha + W\alpha W)\mathrm{D}J ,$$

then V is a G_1-space; and conversely.

Th. 5. *If the Kähler field has the form*

(8) $$K = -\alpha W\alpha\gamma \mathrm{D}J + (\alpha W\alpha + W\alpha W)\mathrm{D}J ,$$

then V is a G_2-space; and conversely.

Th. 6. *If the Kähler field has the form*

(9) $$K = -\gamma \mathrm{D}J + (\alpha W\alpha + W\alpha W)\mathrm{D}J , \quad K = -\gamma \mathrm{D}J - (\alpha W\alpha + W\alpha W)\mathrm{D}J ,$$

then V is respectively a Hermite manifold, an almost Kōto manifold; and conversely.

Th. 7. *If the Kähler field has the form*

(10) $$K = -3\gamma \mathrm{D}J ,$$

then V is an almost Tachibana manifold; and conversely.

Th. 6 shows again that there exists a sort of symmetry between Hermite manifolds and almost Kōto manifolds.

7 - Proofs

To prove Th. 4, remark first that by (6) of $[\mathbf{5}]_4$ relation (7) is equivalent to relation

(11) $$K = -\gamma \mathrm{D}J - 2\alpha \mathrm{D}J + W\alpha \mathrm{D}J + \alpha W\alpha \mathrm{D}J .$$

Comparing this expression with the general expression of K given by (3) in Sec. **3**, since $\alpha\alpha = 1$ and $2\sigma = 1 + \alpha$ we get $\sigma(1 - \alpha W\alpha)\mathrm{D}J = 0$. Therefore V is a G_1-space.

Conversely, we may write the Kähler field K, defined in Sec. **3**, in the form

$$K = (1 + \alpha)\mathrm{D}J - 2\alpha\mathrm{D}J - \gamma\mathrm{D}J\,.$$

On the other hand, if V is a G_1-space, we have

$$(1 + \alpha)\mathrm{D}J = (1 + \alpha)\alpha W\alpha\mathrm{D}J\,.$$

From these equations, since $\alpha\alpha = 1$ we derive equation (11), that is equivalent to relation (7).

So Th. 4 is proved.

Now, taking into account (3) of Sec. **3** and (6) of [**5**]$_4$, we immediately see that relation (8) is equivalent to relation (18) of [**5**]$_4$. Since this latter relation defines G_2-spaces, Th. 5 is proved.

To prove Th. 6, we recall that if V is a Hermite manifold, then V is also a G_2-space [14]. On the other hand, since Hermite manifolds can be defined by condition **a** of Sec. **5** [15], we have $\gamma\mathrm{D}J = \gamma\alpha W\alpha\mathrm{D}J$. Therefore, by proposition P_1 of [**5**]$_4$, Sec. **2**, equation (8) reduces to equation $(9)_1$.

Conversely, comparing (3) with $(9)_1$ and remembering that $\alpha\alpha = 1$, $W\mathrm{D}J = -\mathrm{D}J$, $2\varepsilon = 1 - \alpha$ [16], we have $\varepsilon\mathrm{D}J = \varepsilon\alpha W\alpha\mathrm{D}J$. We can conclude now that V is a Hermite manifold by virtue of Th. 2 of [**5**]$_4$.

We are going now to prove the second part of Th. 6.

If V is an almost Kōto manifold, then condition **b** of Sec. **5** is satisfied. So, using again $(1)_1$, (6) of [**5**]$_4$, from equation (3) we derive $(9)_2$.

Conversely, if the Kähler field K has the form $(9)_2$, using relations (1), (3), (6) of [**5**]$_4$, we have

$$\begin{aligned}\alpha W\alpha WK &= -\alpha W\alpha W\gamma\mathrm{D}J - \alpha W\alpha W\alpha W\alpha\mathrm{D}J - \alpha W\alpha WW\alpha W\mathrm{D}J \\ &= \gamma W\gamma\gamma\mathrm{D}J - W\alpha W\alpha\alpha W\alpha\mathrm{D}J + \alpha W\alpha WW\alpha\mathrm{D}J \\ &= -\gamma\mathrm{D}J + \mathrm{D}J - \alpha\mathrm{D}J\,.\end{aligned}$$

Therefore, bu virtue of (3) in Sec. **3** we can write $\alpha W\alpha WK = K$. So K satisfies condition ***C*** of [**5**]$_2$. We make use now of Sawaki's Lemma [17] and conclude that also $\mathrm{D}J$ satisfies condition ***C***. Consequently, V is an almost Kōto manifold [18].

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(14) See A. Gray and L. M. Hervella [**2**], p. 40.

(15) See footnote (7).

(16) See G. B. Rizza [**5**]$_4$, Sec. **2**, **3**.

(17) This form of Sawaki's Lemma is due to S. Donnini [**1**]. Lemma L, p. 489.

(18) G. B. Rizza [**5**]$_2$, Theorem 5, p. 474.

Finally, if V is an almost Tachibana manifold, then $K = 3\mathrm{D}J$ (19). So from Th. 1 of Sec. **4** we derive relation (10). Conversely, using relation $(5)_2$ of the present paper and relation $(1)_1$ of $[\mathbf{5}]_4$, from equation (10) we derive

$$K = -\gamma K = 3\mathrm{D}J\,. \tag{12}$$

Now, comparing (12) with (10) we get relation (6). Therefore, V is an almost Tachibana manifold by Th 1 of Sec. **4**.

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Sommario

Questo lavoro mostra come alcune note classi di varietà quasi hermitiane possano essere caratterizzate in corrispondenza a particolari espressioni del campo K, associato al differenziale della forma di Kähler. Altre caratterizzazioni, di tipo diverso, vengono pure segnalate.

(19) See K. Yano [**6**], Theorem 1.3, p. 177.

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