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Remarks on the Boltzmann equation for electrons in a gas in electric and magnetic fields (**)

A Luigi Caprioli per il suo 70° compleanno

1 - Introduction

In this note we are concerned in the problem of the motion of a swarm of electrons (mass m, charge $-e$, number density n) in a background gas of atoms (mass M, number density N) in equilibrium at a given temperature T, under the action of an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. Our aim is to throw light on the validity of some truncation procedures of solution of the Boltzmann equation governing the electron distribution function. A particular attention is turned to a perturbation method of solution suggested by I. B. Bernstein [1] and to the meaning of certain equations.

As known, the Boltzmann equation relevant to a swarm of electrons under the above conditions is

$$(1)\qquad \frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial t} + \boldsymbol{v}\cdot\nabla f(\boldsymbol{r}, \boldsymbol{v}, t) - \frac{e}{m}\left[\boldsymbol{E}(\boldsymbol{r}, t) + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r})\right]\cdot\nabla_v f(\boldsymbol{r}, \boldsymbol{v}, t) = J\big(f(\boldsymbol{r}, \boldsymbol{v}, t)\big)\,.$$

In eq. (1), ∇ and ∇_v are the gradient operators in the position and in the velocity

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space, respectively, and J is the so-called collision integral. If we suppose $n \ll N$, in order that electron-electron collisions can be neglected, and we consider only elastic collisions, J is a linear integral operator.

The conventional method of solution of eq. (1) is based on a truncation of the development of f in spherical harmonics in the velocity space, neglecting the terms which are of sufficiently high order with respect to a small parameter, characteristic of the problem (see, for example [**2**], [**3**] and references quoted therein). It is clear, however, that such a truncation presupposes an a-priori evaluation of the coefficients of the spherical harmonics expansion with respect to the small parameter.

For these reasons, it may be suitable to start with a different development, just suggested by Bernstein, under the assumptions

$$\tau_m |\partial \log f/\partial t| \ll 1 , \qquad |eE\tau_m/m| \ll 1 , \qquad v\tau_m/L \ll 1 . \tag{2}$$

In (2) $\tau_m \equiv \tau_m(v)$ is the mean time-of-flight for momentum transfer relevant to an electron of speed v, i.e.

$$\frac{1}{\tau_m(v)} \equiv \nu_m(v) = Nv\int_\Omega \sigma(v, \chi)(1 - \cos\chi)\, d\Omega = 2\pi\int_0^\pi q(v, \chi)(1 - \cos\chi) \sin\chi \, d\chi, \tag{3}$$

where $\sigma(v, \chi)$ is the elastic differential cross section; moreover, L is a macroscopic scale length and we suppose $|\nabla \log f| \sim (1/L)$. Assumptions (2) presume that the mean energy gain per free path due to the electric field is much less than the thermal energy and also presume situations which change very slowly on the scale of τ_m. Then, if we assume as characteristic small parameter the quantity $\alpha = \sqrt{m/M}$, we can consider the quantities on the left-hand-sides in $(2)_2$ and $(2)_3$ of the first order in α, if the assumption is made that the electron and atom energy are of the same order of magnitude (see [**1**], pp. 135-136); moreover the left-hand-side in $(2)_1$ may be considered of the second order in α (see [**1**], p. 137). As regards the magnetic field $\boldsymbol{B}(\boldsymbol{r})$, it will not be assumed to be small, so that the gyration frequency $\omega_c = eB/(mc)$ is allowed to bear an arbitrary relation to $\tau_m(v)$.

Now, let us seek a solution of eq. (1) of the form

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = f_0(\boldsymbol{r}, \boldsymbol{v}, t) + f_1(\boldsymbol{r}, \boldsymbol{v}, t) + f_2(\boldsymbol{r}, \boldsymbol{v}, t) + \dots , \tag{4}$$

where $f_0, f_1, f_2, \dots$ are respectively of order $0, 1, 2, \dots$ in the parameter α. It is just the development (4) the starting point for our discussion.

2 - Equations for f_1 and f_2

Let us now consider the terms of the lowest order in α in eq. (1). Then the following equation for $f_0(\boldsymbol{r}, \boldsymbol{v}, t)$ is obtained

$$\omega_c \frac{\partial f_0(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial \beta} = \int_{\Omega} q(v, \chi)[f_0(\boldsymbol{r}, \boldsymbol{u}, t) - f_0(\boldsymbol{r}, \boldsymbol{v}, t)]\, \mathrm{d}\Omega, \tag{5}$$

where: β is the longitudinal coordinate in the $\boldsymbol{v}$-space, with the polar axis along $\boldsymbol{B}$; $\boldsymbol{u} = (2\boldsymbol{nn} - \mathbf{I}_2)$, where $\mathbf{I}_2$ is the unit dyadic and $\boldsymbol{n}$ is the unit vector directed along the bisector of the angle between the relative velocities before and after collision; note that $|\boldsymbol{u}| = |\boldsymbol{v}|$ (1).
As shown in [1], it can be seen that each solution of (5) is isotropic in $\boldsymbol{v}$, i.e. $f_0(\boldsymbol{r}, \boldsymbol{v}, t) \equiv f_0(\boldsymbol{r}, v, t)$.

If we now proceed to first order in the small parameter α, taking into account assumptions (2) and following observations, eq. (1) yields

$$(\boldsymbol{a} - \frac{e\boldsymbol{B}}{mc} \times \nabla_v f_1) \cdot \boldsymbol{v} = \int_{\Omega} q(v, \chi)[f_1(\boldsymbol{r}, \boldsymbol{u}, t) - f_1(\boldsymbol{r}, \boldsymbol{v}, t)]\, \mathrm{d}\Omega \,, \tag{6}$$

where

$$\boldsymbol{a}(\boldsymbol{r}, v, t) = [\nabla - \frac{e\boldsymbol{E}(\boldsymbol{r}, t)}{mv} \frac{\partial}{\partial v}]\, f_0(\boldsymbol{r}, v, t) \,. \tag{7}$$

Let us seek a solution of (6) for f_1 of the form

$$f_1(\boldsymbol{r}, \boldsymbol{v}, t) = \boldsymbol{g}_1(\boldsymbol{r}, v, t) \cdot \boldsymbol{v} \,, \tag{8}$$

where $\boldsymbol{g}_1$ is isotropic in $\boldsymbol{v}$.

Substitution of (8) into the r.h.s. of (6), yields

$$\boldsymbol{g}_1(\boldsymbol{r}, v, t) \cdot \int_0^{2\pi} \mathrm{d}\psi \int_0^{\pi} q(v, \chi) \sin \chi v [\sin \chi \cos \psi \boldsymbol{i}_1 + \sin \chi \sin \psi \boldsymbol{i}_2 - (1 - \cos \chi)\, \boldsymbol{i}_3]\, \mathrm{d}\chi \tag{9}$$

$$= - \boldsymbol{g}_1(\boldsymbol{r}, v, t) \cdot \frac{\boldsymbol{v}}{\tau_m} \,,$$

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(1) When writing $\boldsymbol{u} = (2\boldsymbol{nn} - \mathbf{I}_2) \cdot \boldsymbol{v}$, it is intended that $\boldsymbol{nn}$ is a tensor product and the dot, between the parenthesis and $\boldsymbol{v}$, indicated a « dot product » of tensors (see, for instance [7]). The same conventions will be adopted in what follows.

where a spherical coordinate system (v, χ, ψ) in the velocity space is taken, with the polar axis in the direction of $\boldsymbol{v} = v\boldsymbol{i}_3$, so that $\boldsymbol{u} = v(\sin\chi\cos\psi\boldsymbol{i}_1 + \sin\chi\sin\psi\boldsymbol{i}_2 + \cos\chi\boldsymbol{i}_3)$.

Thus, eq. (6) assumes the form

$$[\boldsymbol{a} - \frac{e\boldsymbol{B}}{mc} \times \boldsymbol{g}_1 + \frac{\boldsymbol{g}_1}{\tau_m}]\cdot\boldsymbol{v} = 0 . \tag{10}$$

Since the quantity in square brackets in (10) is a function of $\boldsymbol{v}$ which depends only on $v = |\boldsymbol{v}|$, it must vanish, because (10) must hold for all directions of $\boldsymbol{v}$.

Thus we obtain a solution for the isotropic vector $\boldsymbol{g}_1$ (in terms of f_0) by solving the vector equation

$$\boldsymbol{a} - \frac{e\boldsymbol{B}}{mc} \times \boldsymbol{g}_1 + \frac{\boldsymbol{g}_1}{\tau_m} = 0 . \tag{11}$$

To this end, let us take, for a fixed $\boldsymbol{r}$, a cartesian coordinate system (x_1, x_2, x_3) with the x_3-axis parallel to $\boldsymbol{B}$. Then we easily obtain from (11)

$$\begin{aligned} g_{1,1} &= -\tau_m \frac{a_1 - \omega_c\tau_m a_2}{1 + \omega_c^2\tau_m^2} , \\ g_{1,2} &= -\tau_m \frac{a_2 + \omega_c\tau_m a_1}{1 + \omega_c^2\tau_m^2} , \qquad g_{1,3} = -\tau_m a_3 , \end{aligned} \tag{12}$$

where $(g_{1,1}, g_{1,2}, g_{1,3})$ and (a_1, a_2, a_3) are the component of $\boldsymbol{g}_1$ and $\boldsymbol{a}$, respectively. Equations (12) can be written in the compact form

$$\boldsymbol{g}_1 = -\tau_m \mathbf{M}\cdot\boldsymbol{a} , \tag{13}$$

where $(\boldsymbol{b}^* = \boldsymbol{B}/B)$

$$\mathbf{M} = \boldsymbol{b}^*\boldsymbol{b}^* + \frac{1}{1 + \omega_c^2\tau_m^2}(\mathbf{I}_2 - \boldsymbol{b}^*\boldsymbol{b}^*) + \frac{\omega_c\tau_m}{1 + \omega_c^2\tau_m^2}\boldsymbol{b}^*\times\mathbf{I}_2 . \tag{14}$$

Taking account of (7), we finally have

$$\boldsymbol{g}_1 = -\tau_m \mathbf{M}\cdot[\nabla - \frac{e\boldsymbol{E}}{mv}\frac{\partial}{\partial v}]f_0 . \tag{15}$$

Now we will prove that the difference between a general solution of (6) and $\boldsymbol{g}_1 \cdot \boldsymbol{v}$ must be isotropic in $\boldsymbol{v}$. Denoting by φ such a difference, it must satisfy the equation

$$\frac{e\boldsymbol{B}}{mc} \cdot \boldsymbol{v} \times \nabla_v \varphi = \int_\Omega q(v, \chi)[\varphi(\boldsymbol{r}, \boldsymbol{u}, t) - \varphi(\boldsymbol{r}, \boldsymbol{v}, t)]\, \mathrm{d}\Omega\,, \tag{16}$$

that is

$$\omega_c \frac{\partial \varphi}{\partial \beta} = \int_\Omega q(v, \chi)[\varphi(\boldsymbol{r}, \boldsymbol{u}, t) - \varphi(\boldsymbol{r}, \boldsymbol{v}, t)]\, \mathrm{d}\Omega\,. \tag{17}$$

This equation is completely analogous to eq. (5) for f_0 and, in the same way [**1**], it can be seen that each φ must be isotropic in $\boldsymbol{v}$. So we have that the most general form of $f_1(\boldsymbol{r}, \boldsymbol{v}, t)$ is

$$f_1(\boldsymbol{r}, \boldsymbol{v}, t) = \boldsymbol{g}_1(\boldsymbol{r}, v, t) \cdot \boldsymbol{v} + f_1^0(\boldsymbol{r}, v, t)\,, \tag{18}$$

where, therefore, $\boldsymbol{g}_1$ and f_1^0 are an isotropic vector and an isotropic scalar function in the $\boldsymbol{v}$-space, respectively.

If we now proceed to second order in α, eq. (1) yields

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 - \frac{e\boldsymbol{E}}{m} \cdot \nabla_v f_1 - \frac{e\boldsymbol{B}}{mc} \cdot \boldsymbol{v} \times \nabla_v f_2 \tag{19}$$

$$= \int_\Omega q(v, \chi)[f_2(\boldsymbol{r}, \boldsymbol{u}, t) - f_2(\boldsymbol{r}, \boldsymbol{v}, t)]\, \mathrm{d}\Omega + \frac{m}{M} \frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{v^3}{\tau_m} \left(f_0 + \frac{kT}{mv} \frac{\partial f_0}{\partial v}\right)\right].$$

Let us seek, in this case, a solution of the form

$$f_2(\boldsymbol{r}, \boldsymbol{v}, t) = \mathbf{g}_2(\boldsymbol{r}, v, t) : (\boldsymbol{v}\boldsymbol{v})\,, \tag{20}$$

where $\mathbf{g}_2$ is a symmetric tensor of rank 2, whose trace is zero (i.e. $\sum_{k=1}^{3} g_{kk} = 0$) and the symbol « : » indicates the double scalar product between $\mathbf{g}_2$ and $(\boldsymbol{v}\boldsymbol{v})$ (i.e. $\mathbf{g}_2 : (\boldsymbol{v}\boldsymbol{v}) = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} v_j v_i$) [7]. Then, from (19) and (20), we obtain, taking account of (18),

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla (f_1^0 + \boldsymbol{g}_1 \cdot \boldsymbol{v}) - \frac{e\boldsymbol{E}}{m} \cdot \nabla_v (f_1^0 + \boldsymbol{g}_1 \cdot \boldsymbol{v}) - \frac{e\boldsymbol{B}}{mc} \cdot \boldsymbol{v} \times \nabla_v \big(\mathbf{g}_2 : (\boldsymbol{v}\boldsymbol{v})\big) \tag{21}$$

$$= \int_\Omega q(v, \chi)\, \mathbf{g}_2(\boldsymbol{r}, v, t) : (\boldsymbol{u}\boldsymbol{u} - \boldsymbol{v}\boldsymbol{v})\, \mathrm{d}\Omega + \frac{m}{M} \frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{v^3}{\tau_m} \left(f_0 + \frac{kT}{mv} \frac{\partial f_0}{\partial v}\right)\right].$$

If we develop both sides of (21) and take account, in particular, that

$$\frac{e\boldsymbol{E}}{m}\cdot\nabla_v(\boldsymbol{g}_1\cdot\boldsymbol{v}) = \left(\frac{e\boldsymbol{E}}{mv}\frac{\partial\boldsymbol{g}_1}{\partial v}\right):(\boldsymbol{v}\boldsymbol{v}) + \frac{e\boldsymbol{E}}{m}\cdot\boldsymbol{g}_1\,, \tag{22}$$

$$-\frac{e\boldsymbol{B}}{mc}\cdot\boldsymbol{v}\times\nabla_v(\mathbf{g}_2:(\boldsymbol{v}\boldsymbol{v})) = \frac{2e}{mc}\boldsymbol{B}\times\mathbf{g}_2:(\boldsymbol{v}\boldsymbol{v})\ (^2)\,, \tag{23}$$

$$\int_\Omega q(v,\chi)\,\mathbf{g}_2(\boldsymbol{r},v,t):(\boldsymbol{u}\boldsymbol{u}-\boldsymbol{v}\boldsymbol{v})\,\mathrm{d}\Omega = -\nu_2(v)\,\mathbf{g}_2:(\boldsymbol{v}\boldsymbol{v})\,, \tag{24}$$

where

$$\nu_2(v) = \int_\Omega q(v,x)[1-P_2(\cos\chi)]\,\mathrm{d}\Omega\ (^3)\,, \tag{25}$$

we can obtain an equation of the form

$$P + \boldsymbol{Q}\cdot\boldsymbol{v} + \mathbf{R}:(\boldsymbol{v}\boldsymbol{v} - \tfrac{1}{3}v^2\mathbf{I}_2) = 0\,, \tag{26}$$

where P, $\boldsymbol{Q}$, $\mathbf{R}$ are, respectively, a scalar, a vector and a 2-tensor expression, which are isotropic in $\boldsymbol{v}$. In particular we have,

$$P \equiv \frac{\partial f_0}{\partial t} + \frac{v^2}{3}\nabla\cdot\boldsymbol{g}_1 - \frac{e\boldsymbol{E}}{m}\cdot\left(\boldsymbol{g}_1 + \frac{v}{3}\frac{\partial\boldsymbol{g}_1}{\partial v}\right) - \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{v^3}{\tau_m}\left(f_0 + \frac{kT}{mv}\frac{\partial f_0}{\partial v}\right)\right]\,, \tag{27}$$

$$\mathbf{R} \equiv \nabla\boldsymbol{g}_1 - \frac{e\boldsymbol{E}}{mv}\frac{\partial\boldsymbol{g}_1}{\partial v} - \frac{1}{3}\left(\nabla\cdot\boldsymbol{g}_1 - \frac{e\boldsymbol{E}}{mv}\cdot\frac{\partial\boldsymbol{g}_1}{\partial v}\right)\mathbf{I}_2 + \frac{2e}{mc}\boldsymbol{B}\times\mathbf{g}_2 + \nu_2\mathbf{g}_2\,. \tag{28}$$

As $\boldsymbol{v}/v$ is a unit vector whose components are the 1-order spherical harmonics in the velocity space and, similarly, the components of $\boldsymbol{v}\boldsymbol{v}/v^2 - (\frac{1}{3})\mathbf{I}_2$ are 2-order spherical harmonics [**5**], because of the orthogonality properties of the spherical harmonics (we recall that 1 is the 0-order spherical harmonic) P, $\boldsymbol{Q}\cdot\boldsymbol{v}$, $\mathbf{R}:(\boldsymbol{v}\boldsymbol{v} - v^2\mathbf{I}_2/3)$ must be separately zero. So, taking account of (28), we can write the expression of $\mathbf{g}_2$ in terms of $\boldsymbol{g}_1$ (see Appendix) and therefore, by eq. (15), in terms of f_0.

(²) We recall that the « cross product » between a vector $\boldsymbol{A}$ and a tensor $\mathbf{B}$ is defined by $\boldsymbol{A}\times\mathbf{B} = -\mathbf{E}:(\boldsymbol{A}\mathbf{B})$, where $\mathbf{E}$ is the Ricci 3-tensor, whose components E_{ijk} are 0 unless $i \neq j \neq k$ and ± 1 for even/odd permutation of i, j, k from the natural 1, 2, 3 order. This holds also for eq. (14).

(³) $P_2(\cos\chi) = (\frac{3}{2})\cos^2\chi - (\frac{1}{2})$ is the second-order Legendre polynomial in $\cos\chi$.

Now that we have found a solution of eq. (19) of the form (20), we can easily prove, in the same way we have followed for f_1, that the difference between a general solution of (15) for f_2 and $\mathbf{g}_2:(\boldsymbol{vv})$ must be isotropic in $\boldsymbol{v}$. Denoting by $\gamma(\boldsymbol{r}, \boldsymbol{v}, t)$ such a difference, it must satisfy the equation

$$\frac{e\boldsymbol{B}}{mc}\cdot\boldsymbol{v}\times\nabla_v\gamma = \int_\Omega q(v, \chi)[\gamma(\boldsymbol{r}, \boldsymbol{u}, t) - \gamma(\boldsymbol{r}, \boldsymbol{v}, t)]\,\mathrm{d}\Omega\,, \tag{29}$$

or

$$\omega_c\frac{\partial\gamma}{\partial\beta} = \int_\Omega q(v, \chi)[\gamma(\boldsymbol{r}, \boldsymbol{u}, t) - \gamma(\boldsymbol{r}, \boldsymbol{v}, t)]\,\mathrm{d}\Omega\,. \tag{30}$$

completely analogous to eqs. (5) and (17). So γ is isotropic in $\boldsymbol{v}$ and we have

$$f_2(\boldsymbol{r}, \boldsymbol{v}, t) = \mathbf{g}_2(\boldsymbol{r}, v, t):(\boldsymbol{vv}) + f_2^0(\boldsymbol{r}, v, t)\,. \tag{31}$$

If we proceed in an analogous way for the higher order terms in the small parameter α from eq. (1) (i.e. retaining, for each $k\geqslant 3$, the k-order terms and seeking for each f_k of eq. (4) a solution of the form $\mathbf{g}_k \underset{k}{:} (\boldsymbol{v}^k)$, where $\mathbf{g}_k$ is a k-order completely symmetric tensor, $(\boldsymbol{v}^k)$ is the k-times tensor product of $\boldsymbol{v}$ by itself and « $\underset{k}{:}$ » indicates the complete scalar product [7]), we can obtain the perturbation solution of the Boltzmann equation (1), to the desired order of approximation.

3 - Basic equation for f_0 and conclusive remarks

In last section we have proved that f_1 and f_2 have, up to an isotropic function in $\boldsymbol{v}$, the forms (8) and (20), respectively, and we can express both these functions in terms of f_0. Then it remains to obtain the basic equation for f_0. If we take account of (27) and (15), we immediately have

$$\frac{\partial f_0}{\partial t} = \frac{1}{v}\Big(\nabla - \frac{e\boldsymbol{E}}{mv}\frac{\partial}{\partial v}\Big)\cdot\Big[\frac{v^3\tau_m}{3}\,\mathbf{M}\cdot\Big(\nabla - \frac{e\boldsymbol{E}}{mv}\frac{\partial}{\partial v}\Big)f_0\Big] + \frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}\Big[\frac{v^3}{\tau_m}\Big(f_0 + \frac{kT}{mv}\frac{\partial f_0}{\partial v}\Big)\Big]\,. \tag{32}$$

Note that eq. (32) coincides with eq. (3-47) of [1], which, however, was obtained in a different way. One can also see that, when neglecting diffusion processes, eq. (32) yields to the well known Fokker-Planck equation (see, for instance [2], [4]). Some properties of eq. (32), together with the appropriate boundary conditions to be required, can be found in ref. [1]. Now we only want to point our remarks on what follows:

(a) under the assumptions (2), the classical method of solution of (1) consisting in the expansion of $f(\boldsymbol{r}, \boldsymbol{v}, t)$ in spherical harmonics, followed by a truncation procedure, appears as a consequence of the expansion in the small parameter $\alpha = \sqrt{m/M}$, without any a-priori evaluation of the order of the coefficients in the spherical harmonics expansion;

(b) if we limit ourselves to considering only terms of order not higher than the second in α (i.e. not higher than the first in the mass ratio m/M), the first and second order coefficients in the spherical harmonics expansion can be identified with the components of $\boldsymbol{g}_1$ and $\mathbf{g}_2$, respectively (see also [5]); thus the knowledge of f_0 allows us to obtain, in the above approximation, the macroscopic quantities connected with the first and second order moments (e.g. drift velocity, pressure tensor);

(c) as regards the isotropic part of f, which coincides with f_0 to the zero order, we may obtain, if necessary, the first, second, etc. order corrections by solving equations that we can write extending to successive orders the procedure followed in **2**.

Appendix. Calculation of $\mathbf{g}_2$ in terms of $\boldsymbol{g}_1$

As said in **2**, we can obtain the espression of the tensor $\mathbf{g}_2$ in terms of $\boldsymbol{g}_1$ starting by the equation

$$\text{(A1)}\qquad [\nabla \boldsymbol{g}_1 - \frac{e\boldsymbol{E}}{mv}\frac{\partial \boldsymbol{g}_1}{\partial v} - \frac{1}{3}(\nabla \cdot \boldsymbol{g}_1 - \frac{e\boldsymbol{E}}{mv}\cdot\frac{\partial \boldsymbol{g}_1}{\partial v})\,\mathbf{I}_2 + \frac{2e}{mc}\boldsymbol{B}\times\mathbf{g}_2 + \nu_2\,\mathbf{g}_2] : (\boldsymbol{v}\boldsymbol{v} - \frac{v^2}{3}\,\mathbf{I}_2) = 0\,.$$

Now, the quantity in square brackets in (A1) is isotropic in $\boldsymbol{v}$ and the components of $\boldsymbol{v}\boldsymbol{v}/v^2 - \mathbf{I}_2/3$ are spherical harmonics of the second order in the $\boldsymbol{v}$-space [5]. As known (cf. [6], p. 256) for each natural number n, there are $2n+1$ spherical harmonics of order n which are linearly independent. Then, if we multiply successively both sides of (A1) by each fundamental second-order spherical harmonic and integrate over the solid angle, by the orthogonality properties of the spherical harmonics we obtain a system of five linear algebraic equations for the components g_{ij} of $\mathbf{g}_2$, to which we must associate the equation $g_{11} + g_{22} + g_{33} = 0$ (zero trace). Since we have supposed that $\mathbf{g}_2$ is symmetric, we have to solve a linear system of six equations in six variables. One may prove that this system has a unique solution, since the determinant of the coefficient matrix is not zero.

To this end, let us take for a fixed $\boldsymbol{r}$, as in **2**, a cartesian coordinate system (x_1, x_2, x_3) with the x_3-axis parallel to $\boldsymbol{B}$. Then if we put

$$\text{(A2)} \qquad \mathbf{A} = -\left[\nabla \boldsymbol{g}_1 - \frac{eE}{mv}\frac{\partial \boldsymbol{g}_1}{\partial v} - \frac{1}{3}\left(\nabla\cdot \boldsymbol{g}_1 - \frac{eE}{mv}\cdot\frac{\partial \boldsymbol{g}_1}{\partial v}\right)\mathbf{I}_2\right], \qquad \boldsymbol{b} = \frac{2e\boldsymbol{B}}{mc},$$

$$\text{(A3)} \qquad \overline{\mathbf{A}} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T), \qquad (A^T_{ij} = A_{ji} \text{ for each } i, j)$$

(note that the trace of $\mathbf{A}$ and $\overline{\mathbf{A}}$ is zero), we obtain the following linear system

$$\text{(A4)} \quad \nu_2 g_{33} = \bar{A}_{33}, \quad 2\nu_2 g_{13} + b g_{23} = 2\bar{A}_{13}, \quad -b g_{13} + 2\nu_2 g_{23} = 2\bar{A}_{23},$$

$$\nu_2 g_{11} - \nu_2 g_{22} + 2b g_{12} = \bar{A}_{11} - \bar{A}_{22}, \quad -b g_{11} + b g_{22} + 2\nu_2 g_{12} = 2\bar{A}_{12}, \quad g_{11} + g_{22} + g_{33} = 0.$$

The determinant of the coefficient matrix of (A4) has the value $-4\nu_2(\nu_2^2 + b^2)(4\nu_2^2 + b^2)$, so that it is always different from zero, since ν_2 is always positive (see eq. (25)). Denoting by $\bar{A}_{ik}$ the components of $\overline{\mathbf{A}}$, we easily obtain the following solution for the g_{ik}'s

$$\text{(A5)} \qquad g_{11} = \frac{(2\nu_2^2 + b^2)\bar{A}_{11} + b^2\bar{A}_{22} - 2b\nu_2\bar{A}_{12}}{2\nu_2(\nu_2^2 + b^2)},$$

$$g_{22} = \frac{b^2\bar{A}_{11} + (2\nu_2^2 + b^2)\bar{A}_{22} + 2b\nu_2\bar{A}_{12}}{2\nu_2(\nu_2^2 + b^2)}, \qquad g_{33} = \frac{\bar{A}_{33}}{\nu_2},$$

$$g_{12} = g_{21} = \frac{b(\bar{A}_{11} - \bar{A}_{22}) + 2\nu_2\bar{A}_{12}}{2(\nu_2^2 + b^2)}, \qquad g_{13} = g_{31} = \frac{4\nu_2\bar{A}_{13} - 2b\bar{A}_{23}}{4\nu_2^2 + b^2},$$

$$g_{23} = g_{32} = \frac{2b\bar{A}_{13} + 4\nu_2\bar{A}_{23}}{4\nu_2^2 + b^2}.$$

Eqs. (A5) can be given the following compact expression

$$\text{(A6)} \qquad \mathbf{g}_2 = \frac{\nu_2\overline{\mathbf{A}} + \boldsymbol{b}\times\overline{\mathbf{A}} - (1/b^2)[\boldsymbol{b}\times(\overline{\mathbf{A}}\cdot\boldsymbol{b})]\boldsymbol{b} - (1/2\nu_2)(\overline{\mathbf{A}}:\boldsymbol{bb})\mathbf{I}_2}{\nu_2^2 + b^2}$$

$$+ \frac{\tfrac{1}{2}[(\overline{\mathbf{A}}\cdot\boldsymbol{b})\cdot\boldsymbol{b}](\boldsymbol{b}\times\mathbf{I}_2) + 2\{\boldsymbol{b}[\boldsymbol{b}\times(\overline{\mathbf{A}}\cdot\boldsymbol{b})] + [\boldsymbol{b}\times(\overline{\mathbf{A}}\cdot\boldsymbol{b})]\boldsymbol{b}\}}{b^2(4\nu_2^2 + b^2)}$$

$$+ \frac{(3/2\nu_2)(\overline{\mathbf{A}}:\boldsymbol{bb})\boldsymbol{bb} + 3\nu_2[(\overline{\mathbf{A}}\cdot\boldsymbol{b})\boldsymbol{b} + \boldsymbol{b}(\overline{\mathbf{A}}\cdot\boldsymbol{b})]}{(\nu_2^2 + b^2)(4\nu_2^2 + b^2)}.$$

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Sunto

Si discutono i limiti di validità di alcuni metodi di soluzione dell'equazione di Boltzmann per una nube di elettroni in moto in un gas di fondo, costituito di particelle monoatomiche, sotto l'azione di un campo elettrico e di un campo magnetico.

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