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On autodistributive near-rings (**)

Introduction

A groupoid is said to be *autodistributive* if it satisfies the identities $x(yz) = (xy)(xz)$, $(xy)z = (xz)(yz)$. Groupoids with this property have been studied by many Authors (see for ex. [**1**], [**6**], [**10**]). In [**7**] these identities are considered for linear algebras on a field, in [**8**] for semigroups and rings, in $[\mathbf{4}]_1$ for near-rings.

In this paper we continue the study of autodistributive near-rings, showing that such a N has an ideal I with $I^3 = 0$ and N/I idempotent. This fact draw our attention to autodistributive idempotent near-rings: we describe their structure, proving that they are exactly the β-near-rings studied in [**7**].

This paper insert himself into a series of researches on near-rings satisfying a given set of identities. See for ex. $[\mathbf{4}]_2$.

1 - Preliminaries

Throughout this paper the letter N denotes always a non-trivial near-ring (i.e., N cannot be a O-near-ring or a constant one) and we use without explicit mention terminology and results of [**9**], except the fact that we work on *left* near-rings, and we make the obvious change of notation.

Given $a \in N$, we denote by $A_N(a)$ (or simply $A(a)$) the *annihilator* of a in N, that is the set of $x \in N$ such that $ax = 0$; given a subset X of N, we say that X is an *annihilator in* N if there is an $a \in N$ such that $X = A_N(a)$.

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Following [7], a β-near-ring is a near-ring which satisfies the identities $x^2 = x$, $xyz = yxz$.

A near-ring is called *autodistributive* if its multiplicative semigroup is autodistributive. As observed in $[4]_1$, *if N is autodistributive every annihilator in N is an ideal of N*; moreover, *every product of three elements of N is idempotent. In what follows, we use these two facts without explicit mention.*

About these near-rings, we can prove the

Proposition 1. *If the prime number p divides the order of a finite near-ring, in which every product of three elements is idempotent, then the near-ring contains a sub-near-ring of order p.*

Proof. By $[3]_2$, th. 1, it suffices to show that there isn't a p-singular near-ring P in which any product of three elements is idempotent. In fact in P every non-divisor of zero is a product of three elements ($[3]_2$, cor. 6) and these ones cannot be all idempotent (by $[3]_2$ again).

A chain $I_1 \lhd I_2 \lhd ... \lhd I_{m-1} \lhd I_m = N$ (with $m \neq 1$) of ideals of N is called, with $[4]_1$, *special* if, for every $k = 1, 2, ..., m-1$, the set of annihilators, in I_{k+2}/I_k is contained in $\{I_{k+1}/I_k, I_{k+2}/I_k\}$ or in $\{I_k/I_k, I_{k+2}/I_k\}$. A near-ring is said to be *special* if it has a special chain, with $m > 2$, such that I_1 is a 0-near-ring and N/I_{m-1} has not proper annihilators. For shortly, we call *iperspecial* such a chain.

Example 1. *Every near-ring N with $N^3 = 0$ is special.* Namely, the ideal I of N generated by N^2 is a 0-near-ring. Since N is non-trivial, we have $I \neq N$.

If I is not a maximal ideal of N, let J be a proper ideal of N containing I: the chain $I \lhd J \lhd N$ is iperspecial, so N is special.

Otherwise, I is a maximal ideal of N. Let $a \in N$: we have $I \subseteq A_N(a)$ and, since $A_N(a)$ is an ideal of N (because N is of course autodistributive), we have $A_N(a) = I$ or $A_N(a) = N$. So the set of annihilators in $N/0$ is contained in $\{N/0, I_k/0\}$: since 0 and N/I are both 0-near-rings, this proves that the chain $0 \lhd I \lhd N$ is iperspecial, so N is special.

2 - On autodistributive near-rings

We start with the

Theorem 2. *Let N be an autodistributive near-ring. The set I of all nilpotent elements of N is an ideal of N; moreover, $I^3 = 0$ and N/I is idempotent.*

Before the proof, we observe that by this theorem one can easily prove the Cor. 10 of $[4]_1$ (1).

Proof. By $[4]_1$, Oss. 5, every product xya with $x, y \in N$, $a \in I$ is zero. Given $a, b \in I$, $x, y \in N$, we have

$$(a-b)^3 = (a-b)^2a - (a-b)^2b = 0 - 0 = 0\ ;$$

moreover

$$(x+a-x)^3 = (x+a-x)^2x + (x+a-x)^2a - (x+a-x)^2x$$

$$= (x+a-x)^2x - (x+a-x)^2x = 0\ .$$

Thus, from the additive point of view, I is a normal subgroup of N. Since $(xa)^3 = (xa)^2xa = 0$, for $c = (x+a)y - xy$ we have

$$[(x+a)y-xy]^3 = c^2(x+a)y - c^2xy = (c^2x + c^2a)y - c^2xy = c^2xy - c^2xy = 0$$

and this proves that I is an ideal of N; it is clear that $I^3 = 0$.

To prove the idempotency of N/I, it suffices to show that, for $x \in N$, $x - x^2 \in I$. Namely

$$(x-x^2)^3 = (x-x^2)^2x - (x-x^2)^2x^2 = (x-x^2)^2x - (x-x^2)^2x(x-x^2)^2x\ ;$$

but since $(x-x^2)^2x$ is a product of three elements of N, it is idempotent, so the last term of the preceding chain of equalities is zero. It follows $x - x^2 \in I$ and so the theorem is proved.

Lemma 3. *If N satisfies the identity $xyz = z^3$ and $N_0 \neq N$ is a 0-near-ring, different from 0, then N is special.*

Proof. Since N is autodistributive, we have $N_0 \lhd N$. If N_0 is not a maximal ideal of N, let $J \lhd N$ with $N_0 \subseteq J$: the chain $N_0 \lhd J \lhd N$ is iper-special. We can, therefore, suppose that N_0 is a maximal ideal of N. It is clear that

(1) Which states that a distributive and autodistributive near-ring N has an ideal T such that $T^3 = 0$ and N/T is a boolean ring; when Th. 2 is proved, we state $T = I$ and observe that N/T is distributive and idempotent, then $2x = (2x)^2 = 4x^2$, so $2x = 0$ and N/T must be a ring, then a boolean ring.

We remark that [9], in its mention of this corollary (p. 288), omit the hypothesis that N is distributive.

$I \subseteq N_0$; if $I = N_0$, consider the chain $0 \lhd N_0 \lhd N$. To prove that it is special, it suffices to show that the set of annihilators of N is $\{N_0, N\}$.

For $a, b \in N$, $x \in N_0$ we have $abx = x^3 = 0$, because N_0 is a 0-near-ring. So the elements of the form bx ($x \in N_0$) are in $A_N(a)$: since this one is an ideal of N, we have $I \subseteq A_N(a)$.

Since I is maximal, it must be $A_N(a) = I = N_0$ or $A_N(a) = N$. Then, being N/N_0 a constant near-ring, this proves that the given chain is iperspecial.

Now we turn to the case $I \neq N_0$. We consider the chain $I \lhd N_0 \lhd N$. Take $I + a \in N/I$ and let $I + x$ be an element of its annihilator in N/I. Then $ax \in I$ and so $ax_0 + 0x \in I$, thus (being $ax \in N_0$) $0x \in N_0$: but this implies $x \in N_0$. Conversely, if $x \in N_0$, the $ax \in I$. Therefore $A_{N/I}(I + a) = N_0/I$.

Since I is a 0-near-ring and N/N_0 is a constant near-ring, this proves that the chain $I \lhd N_0 \lhd N$ is iperspecial. The lemma is so completely proved.

Theorem 4. *Let N be a near-ring. The following conditions are equivalent:*

(1) *the near-ring N is autodistributive and special;*

(2) *the near-ring N is autodistributive and $N_0^3 = 0$;*

(3) *the near-ring N is autodistributive and its idempotent elements are constant;*

(4) *in N holds the identity $xyz = z^3$.*

Proof. By [4]$_1$, th. 22, the condition (1) implies (2).

Suppose now (2) valid. If a is an idempotent of N, we have $a = a_0 + 0a$, $a^2 = aa_0 + 0a$, so $a_0 = aa_0$, $aa_0 = a^2a_0$, $a_0 = a^2a_0$, then a_0 is idempotent, as a product of three elements; but by (2) it is nilpotent, thus $a_0 = 0$. It follows $a \in N_0$ and (3) is proved.

Let be (3) true. Since every product of three elements is a constant element, we have, $\forall x, y, z \in N$, $xyz = xyxz = yxz = yzxz = zxz$, so $xyz = zxz$ and, for $x = z$, $zyz = z^3$, and (4) follows.

Finally, let (4) holds; then it is clear that N is autodistributive, and so $N_0 \lhd N$.

If $N_0 = N$, it is $N^3 = 0$ and, as we have seen in the preceding paragraph, the near-ring N is special. It cannot be $N_0 = 0$, because N is nontrivial.

We must only consider the case $N_0 \neq N$. Let $J = A_N(N_0)$: it is easily seen that $J \lhd N_0$. Now, if $J = N_0$, N is special (Lemma 3); thus we suppose $J \neq N_0$.

Now we must show that $J \lhd N_0 \lhd N$ is an iperspecial chain. Given $J + a \in N/J$, for $x \in N$, we have $J + x \in A_{N/J}(J + a)$ iff $ax \in J$. Since yax is constant, we have $yax = 0$ iff $x \in N_0$. Then $ax \in J$ iff $x \in N_0$, and so $A_{N/J}(J + a) = N_0/J$ for every $a \in N$: thus the chain $J \lhd N_0 \lhd N$ is iperspecial. It follows (1). This completes the proof.

Example 2. Let $[G, +]$ a group, let $[G, +, \cdot]$, $[G, +, \circ]$ near-rings on the group $[G, +]$, the first rectangular (2) and the second nilpotent of exponent 3. We suppose also that, for any $x, y \in G$, the product xy is in the annihilator of $[G, +, \circ]$. We can now define a near-ring $N = [G \times G, \oplus, \odot]$ as follows

$$(a, b) \oplus (a', b') = (a + a', b + b') , \qquad (a, b) \odot (a', b') = (aa', a \circ b') \ (^3) .$$

It is easily seen that N satisfies the identity $xyz = z^3$, then (Theorem 4) it is *autodistributive and special.*

3 - On β-near-rings

As in [7], we call *small* a near-ring N which has at least a left identity and, for every element $x \in N$ which is not a left identity, it is $xy = 0y$ for any $y \in N$.

The following lemma is analogous, also in the proof, to lemmas 5.3, 5.4 of [7].

Lemma 5. *Let N be an autodistributive idempotent near-ring. If N is subdirectly irreducible, it is small.*

Proof. We state $K = \{x \in N : A(x) \neq 0\}$ and $A = \cap \{A(x) : x \in K\}$. If $K = N$, since N is subdirectly irreducible and every $A(x)$ is an ideal (because N is autodistributive), we have $A \neq 0$. Then, if w is a non-zero element of A, it is $w = w^2 = 0$, a contradiction. Thus $K \neq N$.

Let now be $x \in N \setminus K$. For $y \in N$ it is $x(xy - y) = 0$, so $xy = y$ since $A(x) = 0$: this proves that x is a left identity of N.

Let z be a non-zero element of N, which is not a left identity. Reasoning as before, one has $A(z) = 0$, so $A \neq 0$. If $w \in A \setminus \{0\}$, we have $xw = 0$ for $x \in K$. If $wy = 0$ for some $y \in N \setminus \{0\}$, then $w \in K$ and $w = w^2 = 0$, a contradiction. It follows $A(w) = 0$ and w is a left identity: then $zy = zwy = 0y$ for every $y \in N$. This proves the lemma.

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(2) These near-rings have been introduced in $[3]_1$: it is clear that they are exactly the near-rings in which $xy = y^2$ holds.

(3) For example, we show that the product is associative. We have $(a, b) \odot [(a', b') \odot (a'', b'')] = (a, b) \odot (a'a'', a' \circ b'') = (aa'a'', a \circ a' \circ b'') = (aa'a'', 0)$; $[(a, b) \odot (a', b')] \odot (a'', b'') = (aa', a \circ b') \odot (a'', b'') = (aa'a'', (aa') \circ b'') = (aa'a'', 0)$ because aa' is in the annihilator of $[G, +, 0]$.

Lemma 6. *Let N be an idempotent small near-ring. Then N is an autodistributive β-near-ring.*

Proof. Given $x \in N$, if x isn't a left identity, we have $x = x^2 = 0x$, so x is constant. If $x, y \in N$ and x is a left identity, it is $xy = y$, then $0xy = 0y$; otherwise x is constant and $xy = 0y$, so $0xy = 0y$: thus $0xy = 0y$ for every $x, y \in N$.

Given $x, y, z \in N$, we show that the following equalities holds

$$xyz = xyxz = xzyz = yxz . \tag{I}$$

If x is not a left identity, it is constant and $xyz = 0yz = 0z$, $xyxz = 0z$, $xzyz = 0zyz = 0z$, and (I) follows.

If y is not a left identity, it is constant and $xyz = yz = 0z$, $xyxz = yxz = 0xz = 0z$, $xzyz = yz = 0z$, $yxz = 0xz = 0z$, and (I) follows again.

Finally, if both x and y are left identities, the (I) trivially holds. This completes the proof.

Theorem 7. *Let N be a near-ring. The following statements are equivalent:*

(1) *the near-ring N is autodistributive and idempotent;*

(2) *the near-ring N is a β-near-ring;*

(3) *the near-ring N is a subdirect sum of a family of subdirectly irreducible near-rings, each of which is idempotent and small.*

Proof. Assume (1). Since N is a subdirect sum of a family of subdcirectly irreducible near-rings, each of which is a factor near-ring of N, to prove (3) it suffices to show that these are small: but this follows from Lemma 5.

Since (Lemma 6) the small and idempotent near-rings are autodistributive, (3) implies (1) and, by Lemma 6 again, also (2).

Thus, being known from [**7**] that (2) implies (3), the assertion follows.

4 - On CL-boolean near-rings

In this paragraph we study a class of autodistributive near-rings, obtained by a construction introduced in [**2**].

In accordance with the definition given in [**2**], an idempotent near-ring N is called here *CL-boolean* if there is a boolean ring $\mathscr{B} = [B, +, \wedge, 1]$ such that $[B, +]$ is the additive group of N and the product defined in N is an algebraic operation on $\mathscr{B}$. Again with [**2**], the near-ring N is called *CL-special-boolean* if its product is defined by $xy = (x \vee b) \wedge y$, where b is a fixed element of a boolean ring $\mathscr{B} = [B, +, \wedge, 1]$.

Theorem 8. *Let $\mathscr{B} = [B, +, \wedge, 1]$ be a boolean ring, let $\cdot$ be a binary algebraic operation on $\mathscr{B}$. The structure $N = [B, +, \cdot]$ is a near-ring iff $xy = (x \wedge c + b) \wedge y$, where $b, c \in B$ and $b \wedge c = 0$. Then N is autodistributive and satisfies the identities $x^3 = x^2$, $xyz = yxz$.*

Proof. It is easy to see that the operation $\cdot$ is defined by

$$xy = u + x \wedge a + y \wedge b + x \wedge y \wedge c\,, \qquad \text{with } u, a, b, c \in B\,.$$

Let suppose that N is a near-ring. Thus $u = 0$; moreover $0 = x0 = x \wedge a$ for every $x \in B$, then $a = 0$. Hence $xy = y \wedge b + x \wedge y \wedge c = y \wedge (b + x \wedge c)$.

Now $(xy)z = b \wedge y \wedge c \wedge z + c \wedge x \wedge y \wedge z + b \wedge z$ and $x(yz) = b \wedge z + b \wedge c \wedge y \wedge z + c \wedge x \wedge b \wedge z + c \wedge x \wedge y \wedge z$, then $c \wedge x \wedge b \wedge z = 0$ for every $x, z \in B$ and so $b \wedge c = 0$.

Conversely, it is easy to show that every operation satisfying these conditions gives an autodistributive near-ring on $[B, +]$, in which $x^3 = x^2$ and $xyz = yxz$ holds.

We remark that in a near-ring N as before it is $xyz = c \wedge x \wedge y \wedge z + b \wedge z$, then, in general, $xyz \neq z^3$: by Theorem 4, in general N is *not* special.

Clay and Lawver posed the problem of classify the CL-boolean near-ring ([4]). We solve it by

Corollary 9. *Every CL-boolean near-ring is CL-special-boolean.*

Proof. Let N be a CL-boolean near-ring. By Theorem 8, we have $xy = (x \wedge c + b) \wedge y$, with $b, c \in B$, $b \wedge c = 0$. Since now N is idempotent, $x = x^2 = (x \wedge c + b) \wedge x = x \wedge c + b \wedge x = x \wedge (b + c)$, then $x = x \wedge (b + c)$ for every $x \in B$, so $b + c = 1$. Hence $xy = (x \wedge (b + 1) + b)y = (x \wedge b + x + b)y = (x \vee b) \wedge y$, where $\vee$ is the union defined in the lattice associated to $\mathscr{B}$. This proves the assertion.

([4]) See [**2**], p. 272.

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Riassunto

Studiamo la struttura dei quasi-anelli autodistributivi, mostrando, tra l'altro, che un tale N ha un ideale I tale che $I^3 = 0$ e N/I sia idempotente. Mostriamo anche che i quasi-anelli autodistributivi e idempotenti sono tutti e soli i β-quasi-anelli introdotti da Ligh. Risolviamo inoltre un problema proposto da Clay e Lawver.

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