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The commutativity of certain groups of fixed-point-free automorphisms (**)

If G is a group and H is a nontrivial group of fixed-point-free automorphisms of G, there is a left near integral domain N such that G is the additive group of N and the endomorphisms of G induced by nonzero left multipliers in N are the automorphisms in H [**2**]. If the correspondence from nonzero elements of N onto H is injective, N is a nearfield and G is abelian [**3**]. Let R be a commutative ring with unit element 1. Define G_n to be the multiplicative group of $n\times n$ matrices over R such that a matrix is in G_n precisely if the matrix has the form $I+T$, where I is the identity matrix and T is upper strictly triangular. Define G_∞ to be the analogous group of upper triangular matrices with a row and column indexed by each positive integer. G_n is nilpotent of class $n-1$ for n finite and G_∞ is nonnilpotent. Let t be a unit of R such that if n is infinite, the multiplicative order of t is infinite; or if n is finite, the order of t is either infinite or relatively prime to $(n-1)!$. The automorphisms of G_n induced by conjugation by the diagonal matrix $\sum_{i=1}^{n} t^i E_{ii}$ generates a group of fixed-point-free automorphisms of G_n. Adams [**1**] used these groups to generate examples of near integral domains which have additive groups that can be nonnilpotent or nilpotent of any class. This note shows that all near integral domains with $G=G_n$ and H a group of fixed-point-free automorphisms induced by conjugations of G_n by $n\times n$ matrices must have H isomorphic to a multiplicative group of $n\times n$ diagonal matrices.

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If M is an invertible $n \times n$ matrix, the condition $MG_nM^{-1} \subset G_n$ requires that M be an upper triangular matrix. If $M = \sum_{i,j} a_{ij}E_{ij}$, $1 \leqq i \leqq j \leqq n$, is an invertible triangular matrix, $M^{-1} = \sum_{i=1}^{n} a_{ii}^{-1}E_{ii} + \sum_{1 \leqslant i < j \leqslant n} b_{ij}E_{ij}$ for some elements $b_{ij} \in R$. Hence the diagonal coefficients of M are units in R. Because conjugation by $a_{11}^{-1}M$ is the same as conjugation by M, we can define each automorphism h in H as conjugation by an upper triangular matrix $M = m(h)$ with $a_{11} = 1$. The correspondence $h \to m(h)$ is an isomorphism. For a matrix M let $d(M)$ be the diagonal matrix formed by replacing all off-diagonal elements of M by zeros. If $d(m(h)) = d(m(h^*))$, $m(h^{-1}h^*) = (m(h))^{-1}m(h^*) \in G_n$. Because $h^{-1}h^*$ leaves $m(h^{-1}h^*)$ fixed, $h^{-1}h^*$ is the identity of H. Therefore $h \to d(m(h))$ is an isomorphism of H onto a group of diagonal matrices.

For nearfields G must be abelian. In the matrix examples H must be abelian. If G is a free group and H is a group of fixed-point-free automorphisms induced by a regular permutation group of the generators of G, both G and H can be nonabelian in the same example.

References

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