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An existence analysis for a multipoint boundary value problem via the alternative method (**)

Introduction

Boundary value problems (BVP) of various types are some of the major areas of study in the theory of differential equations. There are many physical and engineering problems where multipoint boundary value problems (MPBVP) arise. The vibrating beam problems with point loadings [**8**] and the difference schemes in numerical analysis [**10**] are some of them. The present paper is a contribution in this direction concerning MPBVP.

In series of papers, Cesari $[\mathbf{1}]_{1,2}$, Locker $[\mathbf{7}]_{1,2}$ and Hale [**6**] developed a process based on functional analysis, for the solution of operational equations of the form $Lx = Nx$, L a linear possibly unbounded operator, N a continuous possibly nonlinear operator, which was denoted in [**6**] as the alternative method. This process was applied by many authors to the solution of nonlinear BVP, both for selfadjoint and nonselfadjoint cases. We only mention here Knobloch, Locker, Osborn and Sather, Landesman and Lazer, Hale, Williams, Mawhin, Kannan. We refer to $[\mathbf{1}]_3$ for references to many of the results in this direction.

Locker studied in detail, in $[\mathbf{7}]_{1,2}$, the case of L a nonselfadjoint linear ordinary differential operator on an interval $[a, b]$ with boundary conditions at a and b, and N a Nemytskii operator not involving derivatives.

In this paper, following Cesari $[1]_{1,2}$ and Locker $[\mathbf{7}]_{1,2}$ we develop an alternative method for the existence analysis of MPBVP for an n-th order differential

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equation. We assume here the boundary conditions are linear and homogeneous, while we allow the nonlinear part to contain derivatives up to and including the derivatives of order $n-1$ of the unknown function. Finally, a numerical problem is indicated, and elsewhere we shall show, by applying the theory, that the problem has indeed a solution.

In a previous paper [**3**]$_1$ we reelaborated the work of Cesari and Locker for nonlinear ordinary differential equations with linear boundary conditions. In another paper [**3**]$_2$ we shall extend the present work for nonlinear ordinary differential equations with multipoint nonlinear conditions.

1 - Notations

In this section, we provide the specific notations used throughout the text.

Here J stands for a closed interval $[a, b]$ on the real line with $b > a$, δ_{ij} denotes the Kronecker delta of i and j, $D(T)$, $N(T)$, $R(T)$ denote the domain, the null space and the range of the operator T, respectively. Also, $\langle\omega_1, \omega_2, \ldots, \omega_m\rangle$ stands for the linear space spanned by $\omega_1, \omega_2, \ldots, \omega_m$, and $T|E$ denotes the restriction of the operator T to the set E. Finally, x_E denotes the characteristic function of E and R^n stands for the n-dimensional real space with Euclidean norm $|\cdot|$.

We shall denote by $x^{(n)}$, or $x^{(n)}(\cdot)$, the n-th derivative of the real-valued function x or $x(\cdot)$, by $x(t+0)$, $x(t-0)$ the right-hand and left-hand limits of x at the point t, respectively, and by $C^n(J)$ the linear space of all n-times continuously differentiable real-valued functions on J. By the Banach space $C^n(J)$ we mean the linear space $C^n(J)$ equipped with the norm $|\cdot|_n$ given by

$$|x|_n = \max_{i=0,1,2,\ldots,n} \sup_{t\in J} |x^{(i)}(t)| \qquad x \in C^n(J)\,.$$

Moreover, $C^\infty(J)$ stands for the linear space of all infinitely differentiable real-valued functions on J and S stands for $L_2(J)$, the Hilbert space of all square-integral real-valued functions on J with the usual inner product and norm denoted by $(\ldots)$ and $\|\cdot\|$, respectively. Also, I denotes the identity operator on S, $E^\perp$ denotes the orthogonal component of E in S, where E is a linear subspace of S. Finally, $E + F$ denotes the direct sum of the subsets E and F of S. For $n \geqslant 1$, the set $H^n(J)$ is defined by

$$H^n(J) = \{x \in C^{n-1}(J)\colon x^{(n-1)} \text{ is absolutely continuous on } J \text{ and } x^{(n)} \in S\}\,.$$

We take $H_0(J) = S$. By the Banach space $H^n(J)$ we mean the linear space $H^n(J)$ equipped with the norm given by

$$|||x||| = \sqrt{b-a}\,\Big(\sum_{i=0}^{n-1} \sup_{t\in J} |x^{(i)}(t)|\Big) + \|x^{(n)}\| \qquad x \in H^n(J)\,, \quad n \geqslant 0\,.$$

The subspace $H_0^n(J)$ and $\tilde{H}^n(J)$ are defined by

$H_0^n(J) = \{x \in H^n(J)\colon x$ and all its derivatives up to and including the order $n-1$ vanish at both end points a and $b\}$ $\quad n \geqslant 1$;
$\tilde{H}^n(J) = \{x \in H^n(J)\colon x^{(n)}$ is essentially bounded$\}$ $\quad n \geqslant 0$.

The norm μ on $\tilde{H}^{n-1}(J)$ is defined by

$$\mu(x) = \max\big(\max_{i=0,1,\dots,n-2} \sup_{t\in J} |x^{(i)}(t)|, \operatorname{ess.sup} |x^{n-1}(t)|\big) \qquad x \in \tilde{H}^{(n-1)}(J), \quad n \geqslant 1 .$$

2 - Formulation of the problem and general assumptions

A sufficiently general MPBVP for an n-th order nonlinear differential equation is of the following form

$$\text{(2.1)} \qquad \tau x = p_n(t)\,\mathrm{d}x^n/\mathrm{d}t^n + p_{n-1}(t)\,\mathrm{d}^{n-1}x/\mathrm{d}t^{n-1} + \dots + p_0(t)x = X(t, x, x^{(1)}, \dots, x^{(n-1)}) ,$$

$$\text{(2.2)} \qquad B_j(x) = \sum_{i=0}^{n-1} \big(\alpha_{0ji}x^{(i)}(a) + \alpha_{1ji}x^{(i)}(a_1) + \dots + \alpha_{hji}x^{(i)}(b)\big) = 0 ,$$

where $a = a_0 \leqslant a_1 \leqslant a_2 \leqslant \dots \leqslant a_{h-1} \leqslant a_h = b$, $\quad j = 1, 2, \dots, k$, $\quad k \leqslant n$.

The following conditions are assumed to be valid throughout the present paper.

(i) Each coefficient function $p_i \in C^\infty(J)$, $i = 0, 1, \dots, n$, and $p_n(t) \neq 0$ on J.

(ii) The nonlinear function $X(t, x_0, x_1, \dots, x_{n-1})$ is defined for $t \in J$ and $|x_i| \leqslant R_i$, $i = 0, 1, \dots, n-1$, where each $R_i > 0$.

(iii) $X(., x_0, \dots, x_{n-1}) \in S$ for each fixed $(x_0, \dots, x_{n-1})$ satisfying $|x_i| \leqslant R_i$, $i = 0, 1, \dots, n-1$.

(iv) There exists a real number $k_0 \geqslant 0$ such that for $|x_i| \leqslant R_i$, $|y_i| \leqslant R_i$, we have

$$\text{(2.3)} \qquad |X(t, x, x_1, \dots, x_{n-1}) - X(t, y_0, y_1, \dots, y_{n-1})| \leqslant k_0\big(\sum_{i=0}^{n-1} |x_i - y_i|\big) \qquad t \in J .$$

(v) $\alpha_{0ji}, \alpha_{1ji}, \dots, \alpha_{hji}$ are real constants such that B_j are linearly independent.

In **3** we shall denote by L the linear operator τ when we associate to it the k linear conditions (2.2). In **3** we shall denote by N the Nemytskii operator defined by $Nx = X(t, x(t), x^{(1)}(t), \ldots, x^{(n-1)}(t))$, so that problem (2.1), (2.2) in **8** will take the operational form $Lx = Nx$.

3 - Definitions and elementary properties of L and H

For the formal differential operator τ under assumptions (ii) and the set of k linearly independent boundary conditions B_j's, the differential operator $L\colon D(L) \subset S \to R(L) \subset S$ is defined as follows

$$D(L) = \{x \in H^n(J)\colon B_j(x) = 0,\ j = 1, 2, \ldots, k\}\,, \qquad Lx = \tau x\,.$$

The operator L has the following properties:

(i) $D(L)$ is dense in S.

(ii) L is a closed linear operator.

(iii) $R(L)$ is closed in S.

(iv) $S = R(L) + N(L^*)$ where L^* denotes the adjoint of L.

(v) $\dim N(L) = p < \infty$ and $\dim N(L^*) = q < \infty$.

Infact, $q \leqslant p \leqslant n$ and $p - q = n - k$.

Proofs of (i), (ii) and (iii) are analogous to the proof of Lemma 7 of Schwartz [**9**], proof of (iv) is clear, and the proof of (v) follows the lines of Theorem 3.4 Chapter 11 of Coddingtion and Levinson [**2**], when one makes use of the variant of the Green's formula for multipoint BVP as given in Wilder [**11**].

We know that the null space of τ is n-dimensional, and that $N(L) \subset N(\tau)$. Let us choose functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in C^\infty(J)$ n solutions of the formal differential operator τ, so that $\varphi_1, \ldots, \varphi_n$ form an orthonormal basis for $N(L)$. We also choose elements $\omega_1, \omega_2, \ldots, \omega_q \in D(L^*)$ to form an orthonormal basis for $N(L^*)$.

We note that the operator $L|D(L) \cap N(L)^\perp$ is a one-to-one closed linear operator having the same range as L. Let H denote the inverse of this operator

$$H = (L|D(L) \cap N(L)^\perp)^{-1}\,. \tag{3.2}$$

By the closed Graph Theorem, H is a one-to-one continuous linear operator.

Cleary, $D(H) = R(L)$ and $R(H) = D(L) \cap N(L)^{\perp}$. Moreover,

$$LHy = y \qquad \text{for all } y \in R(L) , \tag{3.3}$$

$$HLx = x - \sum_{i=1}^{p} (x, \varphi_i)\varphi_i \qquad \text{for all } x \in D(L) . \tag{3.4}$$

Thus H is a continuous right inverse of L.

4 - Projections P_m, Q_m and their relations with L and H

We assume that there exist elements $\omega_{q+1}, \omega_{q+2}, \ldots, \omega_m, \ldots$ belonging to $D(L^*)$ such that the sequence of functions $\omega_1, \ldots, \omega_q, \omega_{q+1}, \omega_{q+2}, \ldots, \omega_m, \ldots$ form a complete orthonormal set in S. Since $S = R(L) + N(L^*)$, the elements $\omega_{q+1}, \omega_{q+2}, \ldots, \omega_m, \ldots$ belong to $R(L)$. Hence $H\omega_{q+i}$, $i \geqslant 1$, are defined and belong to $D(L) \cap N(L)^{\perp}$.

Let

$$S_0 \equiv \langle \varphi_1, \varphi_2, \ldots, \varphi_n, H\omega_{q+1}, \ldots, H\omega_m \rangle . \tag{4.1}$$

We note that $\dim S_0 = p + m - q$. The sequences of projections P_m and Q_m on S are now defined as follows

$$Q_m x = \sum_{i=1}^{m} (x, \omega_i)\omega_i \qquad \text{for all } x \in S , \tag{4.2}$$

$$P_m x = \sum_{i=1}^{p} (x, \varphi_i)\varphi_i + \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i \qquad \text{for all } x \in S , \tag{4.3}$$

where $m > q$. The operators P_m and Q_m have the following properties:

(i) P_m and Q_m are continuous linear opeators defined on all of S.

(ii) $R(Q_m) = \langle \omega_1, \omega_2, \ldots, \omega_m \rangle$.

(iii) $R(P_m) = S_0 \subset D(L)$.

(iv) $P_m^2 = P_m$ and $Q_m^2 = Q_m$.

(v) The range of $I - Q_m$ is a subset of $R(L)$, and $H(I - Q_m)$ is a continuous linear operator defined on all of S.

Theorem 4.1. *The following relations hold:*

(i) $H(I-Q_m)Lx = (I-P_m)x$ *for all* $x \in D(L)$.

(ii) $LH(I-Q_m)x = (I-Q_m)x$ *for all* $x \in S$.

(iii) $LP_m x = Q_m Lx$ *for all* $x \in D(L)$.

(iv) $P_m H(I-Q_m)x = 0$ *for all* $x \in S$.

These identities are immediate consequence of the definitions of the operators L, H, P_m, Q_m, and are essentially the same as those requested by Cesari in $[\mathbf{1}]_{1,2,3}$, and restated by Locker in $[\mathbf{7}]_{1,2}$. Because of these identities, problem $Lx = Mx$ is equivalent to the system of equations

$$x = P_m x + H(I-Q_m)Nx\,, \qquad Q_m(L-N)x = 0\,,$$

as one could verify (cf. Cesari $[\mathbf{1}]_1$). In **9** and in our situation, we shall show in detail that any solution of this system of two equations is also a solution of $Lx = Nx$. In $[\mathbf{1}]_1$ the first of these equations is called the *auxiliary* equation, and the second one the *determining* or *bifurcation* equation.

It may be convenient to consider two copies X and Y of the space S, and note that we have actually performed the decompositions

$$Y = Y_0 + Y_1\,, \qquad Y_0 = \langle \omega_1, \ldots, \omega_m \rangle\,, \qquad Y_1 = Y_0^{\perp}\,,$$

$$X = X_0 + X_1\,, \qquad X_0 = \langle \varphi_1, \ldots, \varphi_n, H\omega_{q+1}, \ldots, H\omega_m \rangle = S_0\,, \qquad X_1 = X_0^{\perp}\,,$$

with $\dim Y_0 = m$, $\dim X_0 = p + m - q$, $H\colon Y_1 \to X_1$, $Q_m\colon Y \to Y$ with $Q_m Y = Y_0$, $(I-Q_m)Y = Y_1$, and $P_m\colon X \to X$, $P_m X = X_0$, $(I-P_m)X = X_1$.

5 - Derivation of certain interval representation for H and $H(I-Q_m)$

Consider the $n \times n$ matrix $\Phi(t)$ which has $\varphi_j^{(i-1)}(t)$ as its entry in the i-th row and j-th column; $i, j = 1, 2, \ldots, n$. For each $t \in J$ this matrix is known to be nonsingular. If we compute Φ^{-1} by forming the adjoint matrix of Φ, then the entry in the j-th row and n-th column of $\Phi^{-1}(t)$ is just $[\det \Phi(t)]^{-1} \cdot W_j(t)$, where $W_j(t)$ is the determinant of the matrix obtained from $\Phi(t)$ by raplacing the j-th column by $(0, 0, \ldots, 1)$. Thus for each $t \in J$, we have

$$\sum_{j=1}^{n} [\det \Phi(t)]^{-1} \Phi_j^{(i)}(t)\, W_j(t) = \begin{cases} 0 & \text{for } i = 0, \ldots, n-2 \\ 1 & \text{for } i = n-1\,. \end{cases} \tag{5.1}$$

Let $G(\cdot,\cdot)$ be the function defined on the square $J\times J$ by

$$(5.2)\qquad G(t, s) = \sum_{i=1}^{n} [p_n(s) \det \Phi(s)]^{-1} \Phi_j(t) W_j(s) \qquad a\leqslant t, s\leqslant b.$$

Clearly $G(\cdot,\cdot)$ is continuous function on $J\times J$ and $G(\cdot, s)\in H^n(J)$.

Lemma 5.1. *Let $y\in S$, and*

$$(5.3)\qquad u(t) = \int_a^t G(t, s) y(s)\,\mathrm{d}s \qquad \text{for all } t\in J\,.$$

Then the function $u\in H^n(J)$ and $\tau u = y$.

This is an immediate consequence of the definitions and of relation (5.1). (The lemma is also stated by Locker in $[\mathbf{7}]_2$ and, in different notations, by Coddington and Levinson in [**2**]).

In the next lemma we obtain a matrix (A_{lj}) which satisfies the equation $(A_{lj})\ (B_j(\varphi_i)) = \hat{I}$, where $\hat{I}$ is the $(n-p)\times(n-p)$ identity matrix.

Lemma 5.2. *There exist real numbers A_{lj}, $l = p+1, \ldots, n$ and $j = 1, 2, \ldots, k$, such that*

$$\sum_{i=1}^{k} A_{lj} B_j(\varphi_i) = \delta_{li} \qquad \text{for } l, i = p+1, \ldots, n\,.$$

Proof. Let B be the $k\times(n-p)$ matrix with entries $B_j(\varphi_i)$, where $j = 1, \ldots, k$ and $i = p+1, \ldots, n$. It is easily seen that B has rank $n-p$. Earlier, we noticed that $k\geqslant n-p$ (see **3** (v)). Consider a suitable $k\times(k-n+p)$ matrix with linearly independent columns and let the matrix be denoted by D. Let $(B{:}D)$ be the $k\times k$ matrix formed by the elements of B and D such that the columns of B occupy the first position. Clearly the matrix D can be choosen such that $(B{:}D)$ is non-singular. Hence $(B{:}D)$ has an inverse. Let the inverse be denoted by A. This is the required matrix.

The following theorem gives an integral representation for H.

Theorem 5.1. *Let $y\in R(L)$. Then Hy has a representation given by*

$$(Hy)(t) = \sum_{l=1}^{n} \varphi_l(t) \int_a^b \varphi_l(s) y(s)\,\mathrm{d}s + \int_a^t G(t, s) y(s)\,\mathrm{d}s \qquad t\in J,$$

where the $\psi_l s$ *are defined as follows*

$$\psi_l(t) = -\int_t^b G(s, t)\Phi_l(s)\,ds \qquad l = 1, 2, \ldots, p\,, \tag{5.4}$$

$$l_l(t) = -\sum_{j=1}^{k}\sum_{i=0}^{n-1}\sum_{r=1}^{n}[p_n(t)\det\Phi(t)]^{-1}A_{lj}W_r(t)\big(\beta_1^{(i)}\chi_{[a,a_1]}(t)+\ldots+\beta_h^{(i)}\chi_{[a_{h-1},b]}(t)\big) \tag{5.5}$$
$$l = p+1, \ldots,\,.$$

Here the constants $\beta_1^{(i)}$ *are defined as follows:*

$$\begin{aligned}
\beta_1^{(i)} &= \alpha_{1ji}\varphi_r^{(i)}(a_1) + \ldots + \alpha_{hji}\varphi_r^{(i)}(a_h)\,,\\
\beta_2^{(i)} &= \alpha_{2ji}\varphi_r^{(i)}(a_2) + \ldots + \alpha_{hji}\varphi_r^{(i)}(a_h)\,,\\
&\ldots\ldots\ldots\ldots\\
\beta_h^{(i)} &= \alpha_{hji}\varphi_r^{(i)}(a_h)\,.
\end{aligned} \tag{5.6}$$

Proof. Take $y \in R(L)$. Let $x = Hy$ and $u(t) = \int_a^t G(t, s)y(s)\,ds$, $t \in J$.

Using basic properties of H together with Lemma 5.1, we notice that $x, u \in H^n(J)$ and $\tau(x-u) = 0$. Hence $x - u \in N(\tau)$. Thus

$$x(t) = \sum_{i=1}^{n} c_i\varphi_i(t) + u(t) \qquad t \in J\,, \tag{5.7}$$

where $c_i s$ are real constants. Since $x \in N(L)^\perp$, we have $(x, \varphi_l) = 0$ for $l = 1, 2, \ldots, p$. But $(x, \varphi_l) = \big(\sum_{i=1}^{n} c_i\varphi_i, \varphi_l\big) + (u, \varphi_l)$. Therefore, $c_l = -(u, \varphi_l)$, $l = 1, 2, \ldots, p$. On the other hand, by Fubini's theorem and by (5.4), we have

$$\begin{aligned}
(u, \varphi_l) &= \int_a^b \Big(\int_a^t G(t, s)y(s)\,ds\Big)\varphi_l(t)\,dt\\
&= \int_a^b \int_s^b G(t, s)y(s)\varphi_l(t)\,dt\,ds = -\int_a^b y(s)\psi_l(s)\,ds\,.
\end{aligned}$$

Thus

$$c_l = -(u, \varphi_l) = \int_a^b y(s)\psi_l(s)\,ds \qquad l = 1, 2, \ldots, p\,. \tag{5.8}$$

Moreover, since $x \in D(L)$, we have $B_j(x) = 0$, $j = 1, 2, \ldots, k$. Since $\varphi_1, \ldots, \varphi_n$

belong to $N(L)$, we have

$$B_j(x) = B_j\Big(\sum_{i=1}^{n} c_i\varphi_i\Big) + B_j(u) = \sum_{i=p+1}^{n} c_i B_j(\varphi_i) + B_j(u) .$$

Thus
$$\sum_{i=p+1}^{n} c_i B_j(\varphi_i) + B_j(u) = 0 \qquad j = 1, 2, ..., k.$$

Multiplying the above equation with A_{lj} and summing up with respect to the index j, we get

$$\sum_{j=1}^{k} \sum_{i=p+1}^{n} c_i A_{li} B_j(\varphi_i) + \sum_{j=1}^{k} A_{lj} B_j(u) = 0 .$$

Hence, by Lemma 5.2, we get

$$c_l = -\sum_{j=1}^{k} a_{lj} B_j(u) \qquad l = p + 1, ..., n .$$

But
$$B_j(u) = \sum_{i=0}^{n-1} \big(\alpha_{0ji} u^{(i)}(a) + ... + \alpha_{hji} u^{(i)}(b)\big).$$

Since $u(t) = \int_a^t G(t, s) y(s)\, ds$, we have

$$u(a) = 0 , \quad u(a_1) = \int_a^{a_1} G(a_1, s) y(s)\, ds, ..., u(b) = \int_a^{b} G(b, s) y(s)\, ds .$$

Further, successive differentiation of u and repeated application of (5.1) yields

$$u^{(i)}(t) = \int_a^t \sum_{r=1}^{n} [p_n(s) \det \Phi(s)]^{-1} \Phi_r^{(i)}(t) W_r(s) y(s)\, ds \qquad i = 0, ..., n-1 .$$

Thus for $i = 0, 1, ..., n-1$, we get

$$u^{(i)}(a) = 0 ,$$

$$u^{(i)}(a_1) = \int_a^{a_1} \sum_{r=1}^{n} [p_n(s) \det \Phi(s)]^{-1} \varphi_r^{(i)}(a_1) W_r(s) y(s)\, ds ,$$

. . . .

$$u^{(i)}(a_{h-1}) = \int_a^{a_{h-1}} \sum_{r=1}^{n} [p_n(s) \det \Phi(s)]^{-1} \varphi_r^{(i)}(a_{h-1}) W_r(s) y(s)\, ds ,$$

$$u^{(i)}(b) = \int_a^{a} \sum_{r=1}^{b} [p_n(s) \det \Phi(s)]^{-1} \varphi_r^{(i)}(b) W_r(s) y(s)\, ds ,$$

Now substituting the above expression in the following, we have, also using (5.6),

$$\sum_{i=0}^{n-1}\big(\alpha_{0ji}u^{(i)}(a)+\ldots+\alpha_{hji}u^{(i)}(b)\big),$$

$$\sum_{i=0}^{n-1}\sum_{r=1}^{n}\big(\alpha_{1ji}\varphi_r^{(i)}(a_1)+\ldots+\alpha_{hji}(\varphi_r^{(i)}(b)\big)\int_a^{a_1}[p_n(s)\det\Phi(s)]^{-1}W_r(s)y(s)\,ds$$

$$+\big(\alpha_{2ji}\varphi_r^{(i)}(a_2)+\ldots+\alpha_{hji}\varphi_r^{(i)}(b)\big)\int_{a_1}^{a_2}[p_n(s)\det\Phi(s)]^{-1}W_r(s)y(s)\,ds$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\alpha_{hji}\varphi_r^{(i)}(b)\int_{a_{h-1}}^{b}[p_n(s)\det\Phi(s)]^{-1}W_r(s)y(s)\,ds\qquad\text{and}$$

$$\sum_{i=0}^{n-1}\sum_{r=1}^{n}\int_a^b[(\beta_1^{(i)}\chi_{[a,a_1)}(s)+\ldots+\beta_h\chi_{(a_{h-1},b]}(s)[p_n(s)\det\Phi(s)]^{-1}W_r(s)y(s)]\,.$$

Substituting this expression in $c_l=-\sum_{j=1}^{k}A_{jl}B_j(u)$, we get

$$c_l=-\sum_{j=1}^{k}\sum_{i=0}^{n-1}\sum_{r=1}^{n}\int_a^b[p_n(s)\det\Phi(s)]^{-1}A_{lj}W_r(s)y(s)\big(\beta_1^{(i)}\chi_{[a,a_1)}(s)+\ldots+\beta_h^{(i)}\chi_{(a_{h-1},b]}(s)\big)\,ds,$$

$l=p+1,\ldots$ and, by (5.5) also

$$c_l=\int_a^b y(s)\psi_l(s)\qquad l=p+1,\ldots,n\,. \tag{5.9}$$

The relations (5.7), (5.8) and (5.9) give the required representation.

Let $K(\cdot,\cdot)$ be the function defined on $J\times J$ by

$$\begin{aligned}K(t,s)&=\sum_{l=1}^{n}\varphi_l(t)\psi_l(s)+G(t,s)\qquad\text{for } a\leqslant s\leqslant t\leqslant b\,,\\K(t,s)&=\sum_{l=1}^{n}\varphi_l(t)\psi_l(s)\qquad\text{for } a\leqslant t\leqslant s\leqslant b\,.\end{aligned}\tag{5.10}$$

We note the following properties of $K(\cdot,\cdot)$:

(i) $K(\cdot,s)$ is continuous on J together with its derivatives up to order $(n-2)$ on J, while the $(n-1$-th derivative $\partial^{-1}K(\cdot,s)/\partial t^{n-1}$ is discontinuous at $t=s$ with a jump given by

$$\partial^{n-1}K(s+0,s)/\partial t^{n-1}-\partial^{n-1}K(s-0,s)/\partial t^{n-1}=1/p_n(s)\,.$$

(ii) For $i = 0, 1, \ldots, n-1$, the function $\partial^i K(t, s)/\partial t^i$ is discontinuous at each of the points $s = a_i$. The discontinuities are of first kind and the jumps are continuous functions of t.

We now state a corollary of Theorem 5.1.

Corollary. *The right inverse operator H has an integral representation given by*

$$(5.11) \qquad (Hy)(t) = \int_a^b K(t, s) y(s)\, \mathrm{d}s \qquad t \in J\,,$$

for all $y \in R(L)$.

Let $K_m(\cdot, \cdot)$ be the function defined on $J \times J$ by

$$(5.12) \qquad K_m(t, s) = K(t, s) - \sum_{i=1}^{m} \Big(\int_a^b K(t, \xi)\, \omega_i(\xi)\, \mathrm{d}\xi \Big)\, \omega_i(s) \qquad a \leqslant t\,, \quad s \leqslant b\,.$$

We notice that $\partial^i K_m(\cdot, \cdot)/\partial t^i$, $i = 0, \ldots, n$, are square integrable on $J \times J$, while the function $\int_a^b (\partial^i K_m(\cdot, \cdot)/\partial t^i)^2 \mathrm{d}s$ $i = 0, \ldots, n-1$, are continuous on J.

The following integral representation for the operator $H(I - Q_m)$ is now an immediate consequence of the above relations.

Theorem 5.2. *Let $x \in S$. Then $\big(H(I - Q_m)x\big)(t) = \int_a^b K_m(t, s) x(s)\, \mathrm{d}s$ $t \in J$.*

6 - Operator equation and assumptions

In the rest of our work we follow the notation of earlier sections and analyse the problem of existence of solutions of the MPBVP

$$(6.1) \qquad Lx = Nx\,,$$

where N is defined as follows

$$(6.2) \qquad \begin{aligned} D(N) = \{x \in \tilde{H}^{n-1}(J) \colon \sup_{t \in J} |x^i(t)| \leqslant R_i \;\; i = 0, \ldots, n-2, \\ \operatorname{ess\cdot sup}_{t \in J} |x^{n-1}(t)| \leqslant R_{n-1}\}\,, \\ (Nx)(t) = X\big(t, x(t), \ldots, x^{(n-1)}(t)\big) \quad \text{for all } t \in J \text{ for which } |x^{(n-1)}(t)| \leqslant R_{n-1}\,. \end{aligned}$$

The following assumptions on L and N are assumed throughout the rest of the work.

(i) L satisfies all the assumptions of **3** and **4**.

(ii) $X(\cdot, \ldots, \cdot)$ satisfies all the assumptions of **2**.

Under the assumptions of **2**, we observe that $Nx \in S$ for $x \in D(N)$.

7 - Some estimates

Let x and $y \in D(N)$. Then, by (2.3) and triangle inequality,

$$\begin{aligned}|||Nx - Ny||| &= |||X(\cdot, x(\cdot), \ldots, x^{(n-1)}(\cdot) - X(\cdot, y(\cdot), \ldots, y^{(n-2)}(\cdot))||| \\ &\leqslant k_0\big(\sum_{i=0}^{n-1} |||x^{(i)} - y^{(i)}|||\big) \\ &\leqslant k_0\big(\sqrt{b-a}\big(\sum_{i=0}^{n-2} \sup_{t\in J} |x^{(i)}(t) - y^{(i)}(t)|\big) + |\,\|x^{(n-1)} - y^{(n-1)}\|\big) \\ &\leqslant k_0 |||x - y|||\,,\end{aligned}$$

where $|||\ |||$ is the norm on $H^{n-1}(J)$. Thus, for $x, y \in D(N)$,

$$\|Nx - Ny\| \leqslant k_0 |||x - y|||\,. \tag{7.1}$$

Let us now define θ_m and $\bar{\theta}_m$ by taking

$$\begin{aligned}\theta_m = \sqrt{b-a}\big(\sum_{i=0}^{n-2} \big(\sup_{t\in J} \int_a^b \big(\partial^i K_m(t, s)/\partial t^i\big)^2 \mathrm{d}s^{1/2}\big) \\ + \int_a^b \int_a^b \big(\partial^{n-1} K_m(t, s)/\partial t^{n-1}\big)^2 \mathrm{d}s\, \mathrm{d}t\big)^{1/2}\,,\end{aligned} \tag{7.2}$$

$$\bar{\theta}_m = \max_{i=0,\ldots,n-1} \sup_{t\in J} \int_a^b \big(\partial^i K_m(t, s)/\partial t^i\big)^2 \mathrm{d}s\big)^{1/2}\,.$$

Noting that $\int_a^b \big(\partial^i K_m(t, s)/\partial t^i\big)^2 \mathrm{d}s$ is a monotone decreasing sequence for every t, Dini's theorem assures uniform convergence of this sequence to zero as $m \to \infty$ for $i = 0, 1, 2, \ldots, n-1$. Thus both θ_m and $\bar{\theta}_m \to 0$ as $m \to \infty$. Let $x \in S$, Then, by Schwarz' inequality and (7.2) we have

$$\|\int_a^b K_m(\cdot, s)x(s)\,\mathrm{d}s\| = \sqrt{b-a}\,[\sum_{i=0}^{n-2} \sup_{t\in J} |\int_a^b \partial^i K_m(t, s)/\partial^i x(s)\,\mathrm{d}s\|]$$

$$+ \|\partial^{n-1} K_m(t, s)/\partial t^{n-1} x(s)\,\mathrm{d}s\|$$

$$\leqslant \sqrt{b-a}\,[\sum_{i=0}^{n-2} \sup_{t\in J} \int_a^b (\partial^i K_m(t, s)/\partial^i)^2\,\mathrm{d}s^{1/2}\|x\|]$$

$$+ \int_a^b \int_a^b (\partial^{n-1} K_m(t, s)/\partial t^{n-1})^2\,\mathrm{d}s\,\mathrm{d}t^{1/2}\|x\| = \theta_m\|x\| .$$

Hence for all $x \in S$ we also have

$$\||\int_a^h K_m(\cdot, s)x(s)\,\mathrm{d}s\|| \leqslant \theta_m \|x\| . \tag{7.3}$$

Similarly, for $x \in S$ we can show that

$$\mu(\int_a^b K_m(\cdot, s)x(s)\,\mathrm{d}s) \leqslant \bar{\theta}_m \|x\| . \tag{7.4}$$

8 - Definition of sets V and $\tilde{S}$

Let us consider the Banach space $H^{n-1}(J)$ with norm $|||\cdot|||$ and pseudo-norm μ. We choose $x_0 \in S_0$ such that $\beta = \mu(x_0) < R$, where $R = \min [R_i,\, i = 0, 1, \ldots, n-1]$. Let $z_0 = H(I - Q_m)Nx_0$, and let e and $\bar{e}$ be real constants such that

$$|||z_0||| \leqslant e , \qquad \mu(z_0) \leqslant \bar{e} . \tag{8.1}$$

Let c, d, r and $\bar{R}$ be positive real numbers such that

$$c + e < d , \qquad \bar{R} + \beta \leqslant R , \qquad r + \bar{e} < \bar{R} . \tag{8.2}$$

The sets V and $\tilde{S}$ in $H^{n-1}(J)$ are defined as follows

$$V = \{x \in S_0 \colon |||x - x_0||| \leqslant c,\, \mu(x - x_0) \leqslant r\} , \tag{8.3}$$

$$\tilde{S} = \{x \in \tilde{H}^{n-1}(J) \colon |||x - x_0||| \leqslant d,\, \mu(x - x_0) \leqslant \bar{R}\} . \tag{8.4}$$

Clearly, $x_0 \in V \subset \tilde{S} \subset D(N)$. Moreover, V and $\tilde{S}$ are closed, bounded and convex subsets of $H^{n-1}(J)$.

We assert that V is a closed and bounded subset of S. Indeed, let $\{y_k\}$ be any sequence contained in V and let $\{y_k\}$ converge to y in the topology of S. We notice that $\{y_k\} \subset H^n(J)$. Since S_0 is finite-dimensional, $y \in S_0$. Also, since linear operators on a finite-dimensional space are bounded, it follows that $\{Ly_k\}$ converges to Ly in the topology of S. Hence, by a lemma of [**5**], the sequence $\{y_k\}$ converges to y in the topology of $H^n(J)$. Hence the sequence $\{y_k\}$ converges to y in the topology of $H^{n-1}(J)$ and $\mu(y_k - y) \to 0$ as $k \to \infty$ and $y \in V$. Obviously, V is a bounded subset of S. This proves our assertion.

9 - Operator T and sets $A(x^*)$ and A

For each $x^* \in V$, let T be the operator on $\tilde{S}$ defined by

$$Tx = x^* + H(I - Q_m)Nx \tag{9.1}$$

for $x \in \tilde{S}$. We observe that T is well defined on S.

For each $x^* \in V$, the set $A(x^*)$ is defined by

$$A(x^*) = \{x \in \tilde{S}: x = Tx\} . \tag{9.2}$$

We denote by

$$A = \bigcup_{x^* \in V} A(x^*) . \tag{9.3}$$

Suppose $A(x^*)$ is non-empty. Then

$$x = Tx = x^* + H(I - Q_m)Nx \quad \text{for some } x \in \tilde{S} . \tag{9.4}$$

Clearly $x \in D(L)$, and by Theorem 4.1 (iv) we have $P_m x = x^*$. Thus

$$Lx = LP_m x + LH(I - Q_m)Nx .$$

Using parts (ii) and (iii) of Theorem 4.1, we get

$$Lx - Nx = Q_m(Lx - Nx) .$$

Hence $x \in \tilde{S}$ is a solution of (6.1), if it satisfies the equation

$$Q_m(Lx - Nx) = 0 , \tag{9.5}$$

Equation (9.4) is the auxiliary equation and (9.5) is the bifurcation equation, as we briefly mentioned in **4**.

In the following section we show that $A(x^*)$ is non-empty. Thus the original MPBVP (6.1) will be reduced to equivalent bifurcation equation (9.5) on the set A.

10 - Reduction of the original MP BVP to an equivalent bifurcation equation and solution of the bifurcation equation

The following theorem with the usual proof reduces the original MPBVP to an equivalent bifurcation equation by making use of the Banach's fixed point theorem.

Theorem 10.1. *Let the assumptions of* **6** *and conditions* (8.1) *and* (8.2) *be valid. Let «m» be sufficiently large such that*

$$\theta_m k_0 < 1 , \quad c + e \leqslant (1 - \theta_m k_0) d , \quad r + \bar{e} \leqslant \bar{R} - \bar{\theta}_m k_0 d . \tag{10.1}$$

Then for each $x^* \in V$ the set $A(x^*)$ is singleton. Moreover, the singleton $A(x^*)$ varies continuously with x^* and $Lx - Nx = Q_m(Lx - Nx)$ on the set A.

By Theorem 10.1, for each $x^* \in V$ there exists a unique element $\hat{x} \in A \subset \tilde{S}$ such that

$$\hat{x} = T\hat{x} = x^* + H(I - Q_m)Nx .$$

Let $\Gamma(x^*) = \hat{x}$. We note that $\Gamma\colon V \to D(L) \cap \tilde{S}$ and is continuous.

The next theorem is an immediate corollary of Theorem 10.1.

Theorem 10.2. *Let the assumptions of Theorem* 10.1 *be valid. Suppose there exists an $x^* \in V$ such that*

$$Q_m(L\Gamma x^* - N\Gamma x^*) = 0 , \tag{10.2}$$

then $\hat{x} = \Gamma x^*$ is a solution of the MPBVP *$Lx = Nx$. Further,*

$$Q_m\hat{x} = x^* , \quad |||\hat{x} - x_0||| \leqslant d , \quad \mu(\hat{x} - \hat{x}_0) \leqslant \bar{R} .$$

Thus, we have found a unique solution $\hat{x} = \Gamma x^*$ of the ausiliary equation $x = x^* + H(I - Q_m)Nx$, and by substitution the bifurcation equation

$Q_m(Lx - Nx) = 0$ has taken the form (10.2). In the terminology of [**1**]$_3$ and [**6**] we may say that we have reduced the original MPBVP problem to the alternative problem (10.2) in the finite dimensional space X_0.

Elsewhere we shall apply the general process discussed above to a numerical problem. Namely we shall show that the problem

$$x''' = (xx')^2 + t - (2/\pi)\sin \pi t\,, \qquad (') = \mathrm{d}/\mathrm{d}t\,, \qquad x'(0) = x'(1) = (1/2) = 0$$

has indeed a solution $x \in H^3(J)$. In this problem we take $J = [0,1]$, $\tau x = x'''$, $B_1(x) = x'(0)$, $B_2(x) = x'(1)$, $B_3(x) = x(1/2)$, and then the operator L is defined by $Lx = \tau x$, $D(L) = [x \in H^3(J), x'(0) = x'(1) = x(1/2)]$.

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