Riv. Mat. Univ. Parma (4) **10** (1984), 123-129

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Linear symplectic relations and floating networks (**)

Introduction

Floating reciprocal networks are analysed. It is shown that such networks can be composed of resistors interconnecting terminals. Results are well known, however the use of simplicial complexes and symplectic relations gives new meaning to the derivations.

Simplicial complexes were used to analyse networks by Oster and Perelson [**4**] and by Smale [**5**]. A general theory of linear symplectic relations was developed by Benenti and Tulczyjew [**1**].

This paper is related to an earlier paper [**2**] in which interconnections of reciprocal networks are studied.

For a general theory of linear networks see [**3**].

1 - Complexes associated with an n-terminal network

We denote by K the set of terminals of an n-terminal network. For each $q = 0, 1, \dots$ we introduce two spaces C^q and C_q: The space C^q is defined as the set of formal combinations

$$a = \frac{1}{(q+1)!}\, a^{\mu_0, \dots, \mu_q} \quad \mu_0, \dots, \mu_q$$

of elements $(\mu_0, \dots, \mu_q)$ of K^{q+1}, with real, totally skewsymmetric coefficients.

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(**) Lavoro eseguito nell'ambito del G.N.F.M. (C.N.R.). — Ricevuto: 26-X-1982.

The space C_q is defined as the set of real skewsymmetric functions on K^{q+1}

$$b: K^{q+1} \to R: (\mu_0, \ldots, \mu_q) \mapsto b_{\mu_0, \ldots, \mu_q} .$$

Spaces C^q and C_q are dual to each other.

A pairing $\langle , \rangle : C_q \times C^q \to R$ is defined by

$$\langle a, b \rangle = \frac{1}{(q+1)!} a^{\mu_0, \ldots, \mu_q} \; b_{\mu_0, \ldots, \mu_q} .$$

Spaces C^q form a chain complex

$$C^0 \xleftarrow{\partial^1} C^1 \xleftarrow{\partial^2} C^2 \xleftarrow{\partial^3} C^3 \xleftarrow{\partial^4} \ldots ,$$

with the operators ∂^q defined by

$$\partial^q\Big(\frac{1}{(q+1)!} a^{\mu_0, \ldots, \mu_q} (\mu_0, \ldots, \mu_q)\Big) = \frac{1}{q!} \sum_\mu a^{\mu\mu_0, \ldots, \mu_{q-1}} \; (\mu_0, \ldots, \mu_{q-1}) .$$

Spaces C_q form a cochain complex

$$C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} C_3 \xrightarrow{d_3} \ldots ,$$

with the operators d_q defined by

$$d_q b(\mu_0, \ldots, \mu_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \; (\mu_0, \ldots, \hat{\mu}_i, \ldots, \mu_{q+1}) ,$$

where $\hat{\mu}_i$ indicates that μ_i has been removed from the sequence $(\mu_0, \ldots, \mu_{q+1})$. Relations $\partial^q \partial^{q+1} = 0$ and $d_{q+1} d_q = 0$ are obviously satisfied.

Both complexes can be augmented. The mappings

$$\eta : C^0 \to R : \sum_\mu a^\mu(\mu) \mapsto \sum_\mu a^\mu , \qquad \varepsilon : R \to C_0 : b \mapsto \varepsilon b ,$$

where $\varepsilon b(\mu) = b$, satisfy $\eta \partial^1 = 0$ and $d_0 \varepsilon = 0$.

Hence, sequences

$$0 \leftarrow R \xleftarrow{\eta} C^0 \xleftarrow{\partial^1} C^1 \xleftarrow{\partial^2} C^2 \xleftarrow{\partial^3} \ldots , \qquad 0 \to R \xrightarrow{\varepsilon} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \ldots$$

are augmented complexes. It can be easily shown that these sequences are exact.

The two augmented exact sequences are dual to each other because operators η, ε, ∂^q and d_q satisfy the duality relations

$$\langle a, \varepsilon b\rangle = \langle \eta a, b\rangle, \ \langle a, d_q b\rangle = \langle \partial^{q+1} a, b\rangle,$$

for any q and any a and b in the appropriate spaces.

The pairing $\langle\ ,\ \rangle : R\times R \to R$ is defined by $\langle a, b\rangle = ab$.

2 - Nodal analysis in linear electric networks

The general problem in electrical circuit theory is to determine how a given network responds a set of excitations. Let us consider a linear electric network, with n nodes and associate to each node μ a nodal voltage v_μ, which is defined as the voltage between node μ and the ground. From physical experience we know that, for usual networks, any set of nodal voltages (excitation) gives rise to a unique set of nodal currents (response). The excitation-response correspondence can be expressed by n linear relations

$$i^\mu = G^{\mu\lambda} v_\lambda \qquad \mu, \lambda = 1, 2, \dots, n\,, \tag{2.1}$$

where the matrix $G^{\mu\lambda}$ characterizes the behaviour of the network. In (2.1) the summation convention holds.

If $\det G^{\mu\lambda} \neq 0$, relations (2.1) can be inverted to obtain

$$v_\lambda = R^{\lambda\mu} i_\mu ; \tag{2.2}$$

this means we can consider a set of nodal currents as an excitation and obtain a unique response in terms of nodal voltages.

In linear network theory it is usually assumed that the matrix $G \equiv (G^{\mu\lambda})$ is symmetric. We shall formulate a geometric counterpart of this condition in a suitable space, the direct sum of the current and voltage spaces.

Let I denote the vector space of nodal currents and let V be the space of nodal voltages. These spaces are identical with the spaces C^0 and C_0 introduced in 1. The space $P_0 = I \oplus V$ has a canonical symplectic structure induced by the duality of I and V. We denote by ω_0 the symplectic 2-form on P_0 defined by

$$\begin{aligned}\omega_0 : P_0 \times P_0 \to R : (i \oplus v, i' \oplus v') &\mapsto \langle i', v\rangle - \langle i, v'\rangle \\ &= \sum_\mu (i'^\mu v_\mu - i^\mu v'_\mu)\,.\end{aligned} \tag{2.3}$$

The relation (2.1), which we can write in the abstract form $i = Gv$, is geometrically represented in P_0 by its graph

$$N_0 = \{i \oplus v \varepsilon P_0;\ i = Gv\} . \tag{2.4}$$

The behaviour of a linear electric network can be represented by a subspace N_0 of P_0 even if relation (2.1) can not be used.

A subspace N of a symplectic space (P, ω) is said to be *Lagrangian* if $\dim N = \frac{1}{2} \dim P$ and the restriction of ω on N vanishes: $\omega_{|N} = 0$. The graph N_0 of G is a Lagrangian subspace of P_0 if and only if G is symmetric. The physical meaning of N_0 being Lagrangian is the network is reciprocal (cfr. [**2**]).

For a reciprocal network the following two conditions are equivalent

$$\text{(i)} \quad \iota_V(\operatorname{im} \varepsilon) \subset N_0 , \qquad\qquad \text{(ii)} \quad \pi_I^{-1}(\ker \eta) \supset N_0 .$$

Here $\iota_V \colon V \to P_0$ is the canonical injection and $\pi_I \colon P_0 \to I$ is the canonical projection.

If N_0 is the graph of a mapping (2.1) then the two conditions can be restricted in the form

$$\text{(i)}' \quad \operatorname{im} \varepsilon \subset \ker G , \qquad\qquad \text{(ii)}' \quad \ker \eta \supset \operatorname{im} G .$$

A reciprocal network is said to be *floating* if the above conditions (i) and (ii) are satisfied. In terms of the matrix $G^{\mu\lambda}$ a network is floating if $\sum_\lambda G^{\mu\lambda} = 0$, for each μ. Floating networks usually satisfy the following conditions

$$\text{(i)}'' \quad \operatorname{im} \varepsilon = \iota_V^{-1}(N_0) , \qquad\qquad \text{(ii)}'' \quad \ker \eta = \pi_I(N_0) ,$$

or

$$\text{(i)}''' \quad \operatorname{im} \varepsilon = \ker G , \qquad\qquad \text{(ii)}''' \quad \ker \eta = \operatorname{im} G .$$

3 - Synthesis of floating reciprocal networks

We introduce the spaces J and E of branch currents and branch voltages. These spaces are identical with the spaces C^1 and C_1, respectively, of **1**.

An element $j = \frac{1}{2} \sum_{\mu,\lambda} j^{\mu\lambda}(\mu, \lambda)$ of J represents the distribution of currents floating inside the network between pairs of terminals. The terminal current distribution $i \in I$ is related to j by

$$i = \partial^1 j \Leftrightarrow i^\mu = \sum_\lambda j^{\lambda\mu} . \tag{3.1}$$

The terminal current distribution i obtained from the relation (3.1) automatically satisfies the constraint $\eta i = 0$.

On the other hand different branch current distributions may produce the same terminal currents.

An element $e \in E$, $e: K \times K \to R: (\mu, \lambda) \mapsto e_{\mu\lambda}$ represents relative voltages between pairs of terminals. These voltages can be obtained from terminal voltages by

$$e = d_1 v \Leftrightarrow e_{\mu\lambda} = v_\lambda - v_\mu . \tag{3.2}$$

Different terminal voltages may produce the same relative voltages and relative voltages e obtained from (3.2) satisfy $d_2 e = 0$.

Spaces J and E are dual to each other. The direct sum $P_1 = J \oplus E$ is a symplectic space. The symplectic form ω_1 is defined by

$$\omega_1: P_1 \times P_1 \to R: (j \oplus e, j' \oplus e') \mapsto \langle j', e\rangle - \langle j, e'\rangle = \tfrac{1}{2} \sum_{\mu,\lambda} (j'^{\mu\lambda} e_{\mu\lambda} - j^{\mu\lambda} e'_{\mu\lambda}) . \tag{3.3}$$

There is a symplectic relation ϱ from P_0 to P_1. Elements $i \oplus v \in P_0$ and $j \oplus e \in P_1$ are in this relation if $i = \partial^1 j$ and $e = d_0 v$. The image $N_1 = \varrho(N_0)$ of the Lagrangian subspace N_0 is a Lagrangian subspace of P_1.

For a floating network the following conditions hold

$$\varrho^t(N_1) = N_0 , \tag{3.4}$$

with ϱ^t the transpose of ϱ, and

(i) $\pi_E^{-1}(\operatorname{im} d_0) \supset N_1$, (ii) $\iota_J(\ker \partial^1) \subset N_1$.

If N_1 is the graph of a mapping $e = rj$, then

(i)′ $\operatorname{im} d_0 \supset \operatorname{im} r$, (ii)″ $\ker \partial^1 \subset \ker r$.

Relation (3.4) implies that N_1 represents equally well as N_0 the behaviour of the network.

Assuming the regularity conditions stated at the end of **2**, we can prove that

(i)″ $\operatorname{im} d_0 = \pi_E(N_1)$, (ii)″ $\ker \partial^1 = \iota_J^{-1}(N_1)$,

or (i)‴ $\operatorname{im} d_0 = \operatorname{im} r$, (ii)‴ $\ker \partial^1 = \ker r$.

An example of the mapping r for a three terminal network is given below

(3.5)$_1$ $$E_{12} = \Delta[(G^{23} + G^{13})J^{12} - G^{13}J^{23} - G^{23}J^{13}]\,,$$

(3.5)$_2$ $$E_{23} = \Delta[(G^{13} + G^{12})J^{23} - G^{12}J^{31} - G^{13}J^{12}]\,,$$

(3.5)$_3$ $$E_{31} = \Delta[(G^{12} + G^{23})J^{31} - G^{23}J^{12} - G^{12}J^{23}]\,,$$

where

$$\Delta = -\frac{1}{G^{12}G^{23} + G^{31}G^{23} + G^{12}G^{13}}\,.$$

This example corresponds to the mapping (2.1), with $\mu, \lambda = 1, 2, 3$, describing N_0 in the case of a three terminal network.

Assuming that N_0 is the graph of the mapping (2.1), there exist mappings $g: E \to J: e \mapsto j = ge$ such that

(3.6) $$G = \partial^1 \circ g \circ d_0\,.$$

Since d_0 is not onto and ∂^1 is not univalent the mapping g is not uniquely determined. We can construct a mapping g distinguished by being diagonal.

From $i^\mu = \sum_\lambda G^{\mu\lambda} v_\lambda$ we obtain

(3.7) $$i^\mu = \sum_\lambda G^{\mu\lambda}(v_\lambda - v_\mu)\,,$$

assuming that $\sum_\lambda G^{\mu\lambda} = 0$. Hence

(3.7)$'$ $$i^\mu = -\sum_\lambda g^{\mu\lambda}(v_\lambda - v_\mu)\,.$$

The mapping $j = ge$ defined by

(3.8) $$j^{\mu\lambda} = -\,g^{\mu\lambda} e_{\mu\lambda}\,,$$

satisfies (3.6) if $g^{\mu\lambda} = -\,G^{\mu\lambda}$ for $\mu \neq \lambda$.

As a result of this construction we have the following synthesyis of a floating reciprocal n-terminal network. The network can be obtained by interconnecting the terminals with resistors; the resistor interconnecting terminals μ and λ has conductance $g^{\mu\lambda}$.

If voltages v_μ are connected to the terminals, then currents $j^{\mu\lambda} = -\,g^{\mu\lambda}(v_\lambda - v_\mu)$ flow between terminals inside the network.

The current $j^{\mu\lambda}$ flow from terminal μ to λ. Hence the total current i^μ flowing into the terminal μ is $i^\mu = \sum_\lambda j^{\mu\lambda} = -\sum_\lambda g^{\mu\lambda}(v_\lambda - v_\mu)$.

Acnowledgement. The Authors are deeply indebted to prof. W. Tulczyjew for his helpful suggestions.

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Summary

N-terminals electrical networks are studied using simplicial complexes and linear symplectic relations. New meaning is given to the derivations of results on floating reciprocal networks.

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