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S. MARCHI and T. NORANDO (*)

Homogenisation estimates for variational inequality of parabolic type (**)

1 - Introduction and results

We set $Y=\prod_{i=1}^{n}[0,y_i]\subset \boldsymbol{R}^n$ $(n>1)$, $\tau_0=[0,k_0]\subset \boldsymbol{R}$, $k_0>0$ fixed. Let $\pi = Y\times\tau_0$; we consider functions $a_{ij}(y,\tau)\in C^2(\pi)$, $i,j=1,2,\dots,n$, such that

$$\sum_{i,j=1}^{n} a_{ij}(y,\tau)\xi_i\xi_j \geqslant \alpha|\xi|^2, \qquad \alpha>0 \qquad \forall\xi\in \boldsymbol{R}^n, \quad \text{a.e. in } \pi .$$

The a_{ij_s} can be extended periodically to $\boldsymbol{R}^n\times\boldsymbol{R}$.

Let us associate the family of operators P^ε to the functions a_{ij} defined by

$$(1.1)_\varepsilon \qquad P^\varepsilon = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\frac{\partial}{\partial x_i}\Big) \qquad (\varepsilon>0),$$

where $x=\varepsilon y$, $t=\varepsilon\tau$. And we set

$$(1.1)_0 \qquad P^0 = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a^0_{ij}\frac{\partial^2}{\partial x_j\partial x_i},$$

where a^0_{ij} are suitable constants such that P^0 is the homogenization operator of the P^ε [**1**].

(*) Indirizzi: S. MARCHI, Istituto di Matematica, Università, 43100 Parma, Italy; T. NORANDO, Istituto di Matematica del Politecnico, Piazza L. Da Vinci 32, 20133 Milano, Italy.

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Let Ω be a bounded open set of $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$ and $T > 0$ fixed, we set $Q = \Omega \times (0, T)$.

For any given function $\psi \in L^2(Q)$, we set $K^\psi = \{v \mid v \in L^2(0, T;\ H_0^1(\Omega)),\ v \leqslant \psi$ a.e. in $Q\}$. We assume $\psi \in C^{\beta,\beta/2}(\bar{Q})$, $u_0 \in C^\beta(\bar{\Omega})$ $(0 < \beta < 1)$.

Let $u^\varepsilon = S^\varepsilon(\psi, u_0)$ be the weak solution, for any $\varepsilon \geqslant 0$, (in the sense of [**5**], [**10**]) of the variational inequality

$$(1.2)_\varepsilon \qquad \begin{gathered} \langle P^\varepsilon u^\varepsilon, v - u^\varepsilon \rangle \geqslant 0 \\ \forall v \in K^\psi, \quad u^\varepsilon \in K^\psi, \quad u^\varepsilon(0) = u_0 . \end{gathered}$$

We have, from $[\mathbf{3}]_3$,

$$(1.3) \qquad \|u^\varepsilon\|_{C^{\beta,\beta/2}(\bar{Q})} \leqslant C \qquad (\varepsilon \geqslant 0),$$

where C does not depend on ε and $\{u^\varepsilon\}_{\varepsilon>0}$ converges weakly to u_0 in $L^2(0, T;\ H_0^1(\Omega))$ for $\varepsilon \to 0$.

We can proove, by the same method used in [**1**] for the elliptic case, that $\{u^\varepsilon\}_{\varepsilon>0}$ converges strongly to u^0 in $L^2(0, T;\ H_0'(\Omega))$ for $\varepsilon \to 0$.

In this paper, we give estimates on the rapidity of convergence in dependence on the smoothness of ψ and u^0. In particular we prove the following results.

Theorem 1. *If we assume, for same* $r > n + 1$, *that* (i) $\|P^\varepsilon \psi\|_{L^r(Q)} \leqslant C$ *for any* $\varepsilon \geqslant 0$, (ii) $\psi_0 = \psi(\cdot, 0) \in W^{2,r}(\Omega)$, (iii) $u_0 \in W^{1,r}(\Omega) \cap H_0^1(\Omega)$, (iv) $\sum_{i,j=1}^n a_{ij}^0 \cdot (\partial^2 u_0/\partial x_i \partial x_j) \in H_0^1(\Omega)$, *then*

$$(1.4) \qquad \|u^\varepsilon - u_0\|_{L^\infty(Q)} \leqslant C \cdot \varepsilon^{\alpha/(n+3\alpha)} .$$

Theorem 2. *If we assume, for some* $r > n + 1$, *that* (i) $\psi \in W^{1,\infty}(0, T;\ W^{-1,r}(\Omega)) \cap L^\infty(0, T;\ W^{1,r}(\Omega))$, (ii) $\psi_0 = \psi(\cdot, 0) \in L^2(\Omega)$, *then we have*

(a) *if* $u_0 \in W^{2,r}(\Omega) \cap H_0^1(\Omega)$,

$$(1.5) \qquad \|u^\varepsilon - u^0\|_{L^\infty(Q)} \leqslant C \cdot \varepsilon^{\alpha/2(n+3\alpha)},$$

(b) if $u_0 \in W_0^{1,r}(\Omega)$,

$$(1.6) \qquad \|u^\varepsilon - u^0\|_{L^\infty(Q)} \leqslant c \cdot \varepsilon^{\alpha/4(n+3\alpha)} .$$

Theorem 3. *If, for some* $\beta (0 < \beta < 1)$, *we have* (i) $\psi \in C^{\beta,\beta/2}(\bar{Q})$, (ii) $u_0 \in C_0^\beta(\bar{\Omega})$, *then*

$$(1.7) \qquad \|u^\varepsilon - u^0\|_{L^\infty(Q)} \leqslant C \cdot \varepsilon^{\alpha\beta/4(n+3\alpha)} .$$

2 - Preliminary results

We consider now, for any $\varepsilon \geqslant 0$, the penalized problem

$$(2.1)_{\varepsilon,\lambda} \qquad P^\varepsilon u_\lambda^\varepsilon + \frac{1}{\lambda}(u_\lambda^\varepsilon - \psi)^+ = 0\,, \qquad u_\lambda^\varepsilon(\cdot, 0) = u_0 \qquad (a > 0)\,.$$

We proove two lemmas.

Lemma 1. *If we assume that, for any* $\varepsilon > 0$ *and for some* $r > n+1$, (i) $\|P^\varepsilon \psi\|_{L^r(Q)} \leqslant C$, *we have*

$$(2.2) \qquad \|P^\varepsilon u_\alpha^\varepsilon\|_{L^r(Q)} \leqslant C \qquad \textit{for some } \varepsilon, \lambda\,,$$

$$(2.3) \qquad \|u_\lambda^\varepsilon - u^\varepsilon\|_{L^r(0,T;\,H_0^1(\Omega))} \leqslant C\lambda^{\frac{1}{2}}\,.$$

Lemma 2. *In the same hypothesis of Lemma* 1, *we have*

$$(2.4) \qquad \|u_\lambda^\varepsilon - u^\varepsilon\|_{L^\infty(Q)} \leqslant C \cdot \lambda^{\alpha/(n+2\alpha)}\,.$$

Proof of Lemma 1. We multiply $(2.1)_{\varepsilon,\lambda}$ by $[(1/\lambda)(u_\lambda^\varepsilon - \psi)^+]^{r-1}$, we obtain

$$\frac{1}{\lambda^{r-1}} \langle P^\varepsilon(u_\lambda^\varepsilon - \psi), [(u_\lambda^\varepsilon - \psi)^+]^{r-1}\rangle + \frac{1}{\lambda^r} \|(u_\lambda^\varepsilon - \psi)^+(t)\|_{L^r(\Omega)}^r$$

$$= \frac{1}{\lambda^{r-1}} \langle - P^\varepsilon \psi, [(u_\lambda^\varepsilon - \psi)^+]^{r-1}\rangle\,.$$

We have

$$\frac{1}{r\lambda^{r-1}} \frac{\mathrm{d}}{\mathrm{d}t} \|(u_\lambda^\varepsilon - \psi)^+(t)\|_{L^r(\Omega)}^r + \frac{1}{\lambda^r} \|(u_\lambda^\varepsilon - \psi)^+(t)\|_{L^r(\Omega)}^r$$

$$\leqslant \frac{1}{\lambda^{r-1}} \|P^\varepsilon \psi(t)\|_{L^r(Q)} \|(u_\lambda^\varepsilon - \psi)^+(t)\|_{L^r(\Omega)}^{r-1}\,,$$

and, by (i), we obtain for any t $(0 < t < T)$

$$\frac{1}{r\lambda^{r-1}} \|(u_\lambda^\varepsilon - \psi)^+(t)\|_{L^r(\Omega)}^r + \frac{1}{\lambda^r} \|(u_\lambda^\varepsilon - \psi)^+\|_{L^r(\Omega\times(0,t))}^r \leqslant \frac{c}{\lambda^{r-1}} \int_0^t \|(u_\lambda^\varepsilon - \psi)^+(s)\|_{L^r(\Omega)}^{r-1} \mathrm{d}s\,.$$

Then we have $\|(u^\varepsilon_\lambda - \psi)^+(s)\|_{L^r(\Omega)} \leqslant c\lambda$ a.e. $s \in (0, t)$ and

$$\frac{1}{\lambda}\|(u^\varepsilon_\lambda - \psi)^+\|_{L^r(Q)} \leqslant C \qquad \text{for any } \varepsilon, \lambda\,, \tag{2.5}$$

and, by (2.5), $(2.1)_{\varepsilon,\lambda}$, (2.2) follows.

We assume now $v = u^\varepsilon_\lambda - (u^\varepsilon_\lambda - \psi)^+$ in $(1.2)_\varepsilon$, (we observe that $u^\varepsilon_\lambda - (u^\varepsilon_\lambda - \psi)^+ \in k^\psi$), and we multiply $(2.1)_{\varepsilon,\lambda}$ by $w - u^\varepsilon_\lambda$, $w \in K^\psi$. We have

$$\langle P^\varepsilon u^\varepsilon, (u^\varepsilon_\lambda - u^\varepsilon) - (u^\varepsilon_\lambda - \psi)^+\rangle \geqslant 0\,, \tag{2.6}$$

$$\langle P^\varepsilon u^\varepsilon_\lambda, w - u^\varepsilon_\lambda\rangle = \frac{1}{\lambda}\langle (u^\varepsilon_\lambda - \psi)^+, u^\varepsilon_\lambda - w\rangle \geqslant 0\,. \tag{2.7}$$

By (2.7), we have in particular

$$\langle P^\varepsilon u^\varepsilon_\lambda, u^\varepsilon_\lambda - u^\varepsilon\rangle \leqslant 0\,. \tag{2.8}$$

By (2.6), (2.8)

$$\langle P^\varepsilon(u^\varepsilon_\lambda - u^\varepsilon), u^\varepsilon_\lambda - u^\varepsilon\rangle + \langle P^\varepsilon u^\varepsilon, (u^\varepsilon_\lambda - \psi)^+\rangle \leqslant 0\,,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u^\varepsilon_\lambda - u^\varepsilon)(t)\|^2_{L^2(\Omega)} + \alpha\|(u^\varepsilon_\lambda - u^\varepsilon)(t)\|^2_{H^1_0(\Omega)} \leqslant \|(P^\varepsilon u^\varepsilon)(t)\|_{L^2(\Omega)}\|(u^\varepsilon_\lambda - \psi)^+(t)\|_{L^2(\Omega)}\,.$$

Then, in virtue of (2.5), (2.2),

$$\int_0^T \|u^\varepsilon_\lambda - u^\varepsilon\|^2_{H^1(\Omega)}(t)\,\mathrm{d}t \leqslant C\lambda\,. \tag{2.9}$$

Proof of Lemma 2. Let α be the De Giorgi-Nash index, dependent on P^ε and r [7], by Lemma 1 we have

$$\|u^\varepsilon_\lambda\|_{C^{\alpha,\alpha/2}(\bar{Q})} \leqslant C\,. \tag{2.10}$$

We consider now the extensions of u^ε_λ and u^ε by zero from Q to $\boldsymbol{R}^{n+1}$. We fix a point $(x_0, t_0) \in Q$ and a real positive number ϱ such that $t_0 + \varrho^2 < T$, $t_0 - \varrho^2 > 0$. We set, for any $t \in (0, T)$,

$$B(\varrho, t) = \{(x, t)\,|\,|x - x_0| < \varrho\}\,,$$

and we consider the mean value of $|u_\lambda^\varepsilon - u^\varepsilon|$ over the cylinder having basis $B_1 = B(\varrho, t_0 - \varrho^2)$ and $B_2 = B(\varrho, t_0 + \varrho^2)$ and height $2\varrho^2$.

$$\frac{1}{2\varrho^2}\int_{t_0-\varrho^2}^{t_0+\varrho^2}\frac{1}{|B(\varrho, t)|}\int_{B(\varrho,t)}|u_\lambda^\varepsilon - u^\varepsilon|\,dx\,dt \leqslant \frac{C}{2\varrho^2}\int_{t_0-\varrho^2}^{t_0+\varrho^2}\|u_\lambda^\varepsilon - u^\varepsilon\|_{L^n(B(\varrho,t))}(t)\,dt$$

$$\leqslant \frac{C}{2\varrho^2}\int_{t_0-\varrho^2}^{t_0+\varrho^2}|B(\varrho, t)|^{(2-n)/(2+n)}\|u_\lambda^\varepsilon - u^\varepsilon\|_{L^2(\Omega)}(t)\,dt$$

$$\leqslant \frac{C}{2\varrho^2}|B(\varrho, t)|^{(2-n)/(2+n)}(2\varrho^2)^{1/2}\|u_\lambda^\varepsilon - u^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C\lambda^{1/2}\varrho^{-n/2}.$$

Then in the cyclinder, by (2.10), we have

$$|u_\lambda^\varepsilon(x, t) - u^\varepsilon(x, t)| \leqslant C\cdot\lambda^{1/2}\varrho^{-n/2} + C\varrho^\alpha, \tag{2.11}$$

and, choosing $\lambda^{1/2} = \varrho^{(n+2\alpha)/2}$, we prove (2.4).

3 - Proof of Theorem 1

We apply the multiple scale method to the problem $(2.1)_{\varepsilon,\lambda}$. We observe that, by hypothesis, we can assume

$$u_\lambda^0 \in L^\infty(0, T;\ W^{2,r}(\Omega) \cap H_0^1(\Omega)). \tag{3.1}$$

From (3.1), (iv), ([**8**] p. 341),

$$u_\lambda^0 \in W_r^{2,1}(Q), \tag{3.2}$$

and, because in this case $P^0\psi \in L^\infty(0, T;\ L^r(\Omega))$,

$$(u_\lambda^0 - \psi)^+ \in W^{1,\infty}(0, T;\ L^r(\Omega)), \tag{3.3}$$

$$(u_0 - \psi_0)^+ = 0. \tag{3.4}$$

Therefore, by (iii),

$$\left\|\frac{\partial u_\lambda^0}{\partial t}\right\|_{W_r^{2,1}(Q)} \leqslant \frac{C}{\lambda}, \tag{3.5}$$

$$\|u_\lambda^0\|_{L^\infty(0,T;\,W^{3,r}(\Omega))} \leqslant \frac{C}{\lambda}. \tag{3.6}$$

We set, as usually

$$P^\varepsilon = \varepsilon^{-2} Q_1 + \varepsilon^{-1} Q_2 + Q_3 \,,$$

where: $Q_1 = A_1$, $Q_2 = \dfrac{\partial}{\partial \tau} + A_2$, $\quad Q_3 = \dfrac{\partial}{\partial t} + A_3$, $\quad A_1 = -\sum_{i,j=1}^{n} [\dfrac{\partial}{\partial y_j} a_{ij}(y, \tau) \dfrac{\partial}{\partial y_i}]$,

$$A_2 = -\sum_{i,j=1}^{n} [\frac{\partial}{\partial y_j} (a_{ij}(y, \tau) \frac{\partial}{\partial x_i}) - \frac{\partial}{\partial x_j} (a_{ij}(y, \tau) \frac{\partial}{\partial y_i})] \,,$$

$$A_3 = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (a_{ij}(y, \tau) \frac{\partial}{\partial x_i}) , \qquad \tau = t/\varepsilon \,, \qquad y = x/\varepsilon \,.$$

We can write: $v^\varepsilon = u_\lambda^0 + \varepsilon \chi_1 + \varepsilon^2 \chi_2$, where $\chi_j = \chi_j(x, y, t, \tau)$ $(j = 1, 2)$ are Y-periodic in y and τ_0-periodic in τ.

We assume $w_\lambda^\varepsilon = u_\lambda^\varepsilon - v^\varepsilon$ and we have

$$\text{(3.7)} \qquad P^\varepsilon w_\lambda^\varepsilon + \frac{1}{\lambda} (u_\lambda^\varepsilon - \psi)^+ - \frac{1}{\lambda} (u^0 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 - \psi)^+$$

$$= - [Q_1 \chi_2 + Q_2 \chi_1 + Q_3 u_\lambda^0 + \frac{1}{\lambda} (u_\lambda^0 - \psi)^+]$$

$$- \varepsilon (Q_3 \chi_1 + Q_2 \chi_2) - \varepsilon^2 Q_3 \chi_2 + \frac{1}{\lambda} (u_\lambda^0 - \psi)^+ + \frac{1}{\lambda} (u_\lambda^0 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 - \psi)^+ \,.$$

We choose now

$$\text{(3.8)} \qquad \chi_1 = -\sum_{i=1}^{n} \theta_i(y, \tau) \frac{\partial u_\lambda^0}{\partial x_i} \,,$$

where $A_1(\theta_i - y_i) = 0$, $\theta_i(y, \tau)$ is Y-periodic in y and τ_0-periodic in τ; then $\theta_i(y, \tau) \in C^{2,2}(Y \times \tau_0)$ and, by (3.7)

$$\text{(3.9)} \qquad \| \chi_1 \|_{C^{2,2}(Y \times \tau_0;\, L^\infty(0,T;\, W^{2,r}(\Omega)) \cap W_r^{1,1}(Q)} \leqslant \frac{C}{\lambda} \,.$$

From (3.8)

$$\text{(3.10)} \qquad \| Q_3 \chi_1 \|_{C^{2,2}(Y \times \tau_0;\, L^r(Q))} \leqslant \frac{C}{\lambda} \,.$$

We choose χ_2 such that the first terme in the second member of (3.7) vanishes. We have

$$Q_1\chi_2 + Q_2\chi_1 + Q_3 u_\lambda^0 + \frac{1}{\lambda}(u_\lambda^0 - \psi)^+ = 0 \,. \tag{3.11}$$

The resolubility condition of (3.11) is given by the equation $(2.1)_{0,\lambda}$. We obtain

$$\|\chi_2\|_{C^{2,1}(y\times\tau_0;\; L^\infty(0,T;\, W^{1,r}(\Omega))\cap W^{1,r}(0,T;\, L^r(\Omega))} \leqslant \frac{C}{\lambda}\,, \tag{3.12}$$

$$\|Q_2\chi_2\|_{C^{2,0}(y\times\tau_0;\; L^\infty(0,T;\, L^r(\Omega))} \leqslant \frac{C}{\lambda}\,. \tag{3.13}$$

We estimate now $Q_3\chi_2$. We observe that

$$\Big\|\frac{\partial\chi_2}{\partial t}\Big\|_{C^{1,0}(y\times\tau_0;\; L^r(Q))} \leqslant \frac{C}{\lambda}\,. \tag{3.14}$$

Moreover we obtain

$$\varepsilon\|A_3\chi_2\|_{L^\infty(0,T;\, W^{-1,r}(\Omega))} \leqslant \frac{C}{\lambda}\,. \tag{3.15}$$

For this purpose, we have, for any $\varphi \in W_0^{1,r'}(\Omega)$, where $1/r' + 1/r = 1$,

$$\Big|\varepsilon \sum_{i,j=1}^{n} \Big\langle a_{ij}(y,\tau)\frac{\partial^2\chi_2}{\partial x_j\,\partial x_i}, \varphi\Big\rangle\Big|$$

$$\leqslant \varepsilon\Big|\sum_{i,j=1}^{n} \int_\Omega \frac{\partial^2\chi_2}{\partial x_j}\Big[\Big(\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(y,\tau)\Big)\varphi(x) + a_{ij}(y,\tau)\frac{\partial\varphi}{\partial x_i}\Big]\,dx \leqslant C\,,$$

a.e. in $t \in \overline{0T}$.

From (3.9) ... (3.15),

$$\|\chi_1(x,y,0,\tau)\|_{C^{2,2}(y\times\tau_0;\; L^\infty(\Omega))} \leqslant \frac{C}{\lambda}\,, \tag{3.16}$$

$$\|\chi_2(x,y,0,\tau)\|_{C^{2,1}(y\times\tau_0;\; L^\infty(\Omega))} \leqslant \frac{C}{\lambda}\,. \tag{3.17}$$

Then we can write (3.7) in the following way

$$(3.18)\qquad P^\varepsilon w_\lambda^\varepsilon + \frac{1}{\lambda}(u_\lambda^\varepsilon - \psi)^+ - \frac{1}{\lambda}(v^\varepsilon - \psi)^+ = \theta_\varepsilon\,, \qquad w_\lambda^\varepsilon(0) = \eta_\varepsilon\,,$$

where

$$(3.19)\qquad \|\theta_\varepsilon\|_{L^\infty(0,T;\,W^{-1,r}(\Omega))} \leqslant \varepsilon\,\frac{C}{\lambda}\,, \qquad \|\eta_\varepsilon\|_{L^\infty(\Omega)} \leqslant \varepsilon\,\frac{C}{\lambda}\,.$$

By the same methods used in [**3**]$_3$, from (3.18), (3.19) we have $\|w_\lambda^\varepsilon\|_{L^\infty(Q)} \leqslant \varepsilon(C/\lambda)$ and therefore

$$(3.20)\qquad \|u^\varepsilon - u^0\|_{L^\infty(Q)} \leqslant C(\lambda^{\alpha/(n+2\alpha)} + \varepsilon\lambda^{-1})\,,$$

choosing $\lambda = \varepsilon^{(n+2\alpha)/(n+3\alpha)}$ in (3.21), we have (1.4).

4 - Proof of Theorem 2

At first, we give a regularization for the obstacle.

We consider, for any η, ε ($\eta > 0$ and $\varepsilon \geqslant 0$), functions which are the parabolic regularizing of ψ, defined by

$$(4.1)\qquad \eta P^\varepsilon \psi_\varepsilon^\eta + \psi_\varepsilon^\eta = \psi\,, \qquad \psi_\varepsilon^\eta(0) = \psi_0\,.$$

It is well know that ψ_ε^η are Hölder continuous [**3**]$_1$.

We make now a change of variable

$$y_i = \sqrt{\eta}\,x_i \quad (i = 1, 2, \dots, n)\,, \qquad \tau = \eta t\,, \quad \eta > 0 \quad \forall (x,t) \in \boldsymbol{R}^{n+1}\,.$$

The operator $P^\varepsilon = P^\varepsilon_{x,t}$ is transformed into an operator $P^\varepsilon_{y,t}$ and we have

$$(4.3)\quad P^\varepsilon_{x,t} = \frac{\partial}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big(a^\varepsilon_{i,j}(x,t)\frac{\partial}{\partial x_i}\Big) = \eta\,\Big[\frac{\partial}{\partial \tau} - \sum_{i,j=1}^n \frac{\partial}{\partial y_j}\Big(a^\varepsilon_{ij}(y,\tau)\frac{\partial}{\partial y_i}\Big)\Big] = \eta P^\varepsilon_{y,\tau}\,.$$

If we consider $\psi = \psi(y,\tau)$, we can read (4.1) in the following way

$$\eta P^\varepsilon_{y,\tau}\psi_\varepsilon^\eta(y,\tau) + \psi_\varepsilon^\eta(y,\tau) = \psi(y,\tau)\,.$$

We set $w_\varepsilon^\eta(y,\tau) = (1/\eta)[\psi_\varepsilon^\eta(y,\tau) - \psi(y,\tau)]$, then

$$P^\varepsilon_{y,\tau}\psi^\eta_\varepsilon(y,\tau) + w^\eta_\varepsilon(y,\tau) = 0\,,$$

$$P^\varepsilon_{y,\tau}[\eta w^\eta_\varepsilon(y,\tau) + \psi(y,\tau)] + w^\eta_\varepsilon(y,\tau) = 0\,,$$

$$P^\varepsilon_{x,t} w^\eta_\varepsilon(x,t) + w^\eta_\varepsilon(x,t) = -P^\varepsilon_{y,\tau}\psi(y,\tau)\,. \tag{4.4}$$

We can apply to (4.4) the L^∞ estimates [7] and we obtain

$$\|w^\eta_\varepsilon(x,t)\|_{L^\infty(Q)} \leqslant \|\psi(y,\tau)\|_{L^\infty(0,T;\,W^{1,r}(\Omega))} \leqslant \eta^{-\frac{1}{2}}\|\psi(x,t)\|_{L^\infty(0,T;\,W^{1,r}(\Omega))}\,, \tag{4.5}$$

and by (4.5)

$$\|\psi^\eta_\varepsilon - \psi\|_{L^\infty(\Omega)} \leqslant C\cdot\eta^{\frac{1}{2}} \qquad \forall\varepsilon \geqslant 0\,, \tag{4.6}$$

$$\|P^\varepsilon\psi^\eta_\varepsilon\|_{L^\infty(\Omega)} \leqslant C\cdot\eta^{-\frac{1}{2}} \qquad \forall\varepsilon \geqslant 0\,. \tag{4.7}$$

We give now a regularization for the initial value u_0.

In the case (a), we define, for any $\eta > 0$, u_0^η by,

$$\eta A^0 u_0^\eta + u_0^\eta = u_0\,. \tag{4.8}$$

It is well know that u_0^η are Holder continuous [3]$_1$, and therefore

$$\|u_0^\eta - u_0\|_{L^\infty(Q)} \leqslant C\cdot\eta^{\frac{1}{2}}, \tag{4.9}$$

$$\|A^0 u_0^\eta\|_{L^\infty(Q)} \leqslant C\cdot\eta^{-\frac{1}{2}}; \tag{4.10}$$

then we have

$$\begin{aligned}\|u^\varepsilon - u\|_{L^\infty(Q)} &= \|S^\varepsilon(\psi,u_0) - S^0(\psi,u_0)\| \\ &\leqslant \|S^\varepsilon(\psi,u_0) - S^\varepsilon(\psi^\eta_\varepsilon,u_0)\| + \|S^\varepsilon(\psi^\eta_\varepsilon,u_0) - S^\varepsilon(\psi^\eta_\varepsilon,u_0^\eta)\| \\ &+ \|S^\varepsilon(\psi^\eta_\varepsilon,u_0^\eta) - S^0(\psi^\eta_\varepsilon,u_0^\eta)\| + \|S^0(\psi^\eta_\varepsilon,u_0^\eta) - S^0(\psi^\eta_\varepsilon,u_0)\| \\ &+ \|S^0(\psi^\eta_\varepsilon,u_0) - S^0(\psi,u_0)\|\,.\end{aligned} \tag{4.11}$$

We observe now that, by Theorem 1 and (4.7), (4.10),

$$\|S^\varepsilon(\psi^\eta_\varepsilon,u_0^\eta) - S^0(\psi^\eta_\varepsilon,u_0^\eta)\| \leqslant C\cdot\varepsilon^{\alpha/(n+3\alpha)}\cdot\eta^{-\frac{1}{2}},$$

and any other terme in the second member of (4.11) is smaller then $C\cdot\eta^{\frac{1}{2}}$ by (4.6), (4.9) and previous considerations.

Choosing in (4.11) $\eta=\varepsilon^{\alpha/(n+3\alpha)}$ we have (1.5).

In the case (b) we apply twice the method used in (a). We obtain at first the regularizing functions in $W^{2,r}(\Omega)\cap H_0^1(\Omega)$ and afterwards in $W^{3,r}(\Omega)$. Then we have in this case

$$\|u_0^\eta-u_0\|_{L^\infty(\Omega)}\leqslant C\cdot\eta^{1/4}, \tag{4.12}$$

$$\|A^0u_0^\eta\|_{L^\infty(\Omega)}\leqslant C\cdot\eta^{-3/4}, \tag{4.13}$$

and so

$$\|S^\varepsilon(\psi_\varepsilon^\eta,u_\varepsilon^\eta)-S^\varepsilon(\psi_\varepsilon^\eta,u_0)\|_{L^\infty(Q)}\leqslant C\cdot\eta^{1/4}\,,$$

$$\|S^\varepsilon(\psi_\varepsilon^\eta,u_0^\eta)-S^0(\psi_\varepsilon^\eta,u_0^\eta)\|_{L^\infty(Q)}\leqslant C\cdot\varepsilon^{\alpha/(n+3\alpha)}\cdot\eta^{-3/4}\,.$$

Then choosing $\eta=\varepsilon^{\alpha/(n+3\alpha)}$ in (4.11) we obtain (1.6).

5 - Proof of Theorem 3

In this case we consider, for any $k>0$, the Friedrichs regularizing functions ψ_k and u_k of ψ and u_0, namely

$$\psi_k(x,t)=\int_{\|z\|\leqslant 1}\int_{|\sigma|\leqslant 1}p(z,\sigma)\,\psi\,(x-\frac{z}{k},\,t-\frac{\sigma}{k})\,\mathrm{d}z\,\mathrm{d}\sigma\,,$$

$$u_k(x)=\int_{\|z\|\leqslant 1}q(z)\,u_0\,(x-\frac{z}{k})\,\mathrm{d}z,$$

where

$$p(z,\sigma)=\begin{cases}C\cdot\exp\left(\dfrac{1}{\|z\|^2+|\sigma|^2-1}\right) & \text{if } \|z\|^2+|\sigma|^2<1\\ 0 & \text{if } \|z\|^2+|\sigma|^2\geqslant 1\,,\end{cases}\qquad \iint p(z,\sigma)\,\mathrm{d}z\,\mathrm{d}\sigma=1,$$

$$q(z)=\begin{cases}C\cdot\exp\left(\dfrac{1}{\|z\|^2-1}\right) & \text{if } \|z\|<1\\ 0 & \text{if } \|z\|\geqslant 1\,,\end{cases}\qquad \int q(z)\,\mathrm{d}z=1\,.$$

We have $\psi_k \in C_0^\infty(Q)$ and $u_k \in C_0^\infty$ and we know that

$$(5.1) \qquad \begin{aligned} &\|\psi_k - \psi\|_{L^\infty(Q)} \leqslant C\cdot k^{-\beta}, \qquad \|\psi_k\|_{W^{1,\infty}(Q)} \leqslant C\cdot k^{1-\beta}, \\ &\|u_k - u_0\|_{L^\infty(\Omega)} \leqslant C\cdot k^{-\beta}, \qquad \|u_k\|_{W^{1,\infty}(\Omega)} \leqslant C\cdot k^{1-\beta}. \end{aligned}$$

Then we obtain

$$(5.2) \quad \begin{aligned} \|u^\varepsilon - u\|_{L^\infty(Q)} &= \|S^\varepsilon(\psi, u_0) - S^0(\psi, u_0)\|_{L^\infty(Q)} \\ &\leqslant \|S^\varepsilon(\psi, u_0) - S^\varepsilon(\psi_k, u_0)\|_{L^\infty(Q)} + \|S^\varepsilon(\psi_k, u_0) - S^\varepsilon(\psi_k, u_k)\|_{L^\infty(Q)} \\ &+ \|S^\varepsilon(\psi_k, u_k) - S^0(\psi_k, u_k)\|_{L^\infty(Q)} + \|S^0(\psi_k, u_k) - S^0(\psi_k, u_0\|_{L^\infty(Q)} \\ &+ \|S^0(\psi_k, u_0) - S^0(\psi, u_0)\|_{L^\infty(Q)}. \end{aligned}$$

We observe that, from Theorem 2 and (5.1)

$$\|S^\varepsilon(\psi_k, u_k) - S^0(\psi_k, u_k)\|_{L^\infty(Q)} \leqslant C\cdot \varepsilon^{\alpha/4(n+3\alpha)}\cdot k^{1-\beta}$$

and any other terme in the second member of (5.2) is smaller then $C\cdot k^{-\beta}$. Choosing $k^{-1} = \varepsilon^{\alpha/4(n+3\alpha)}$ in (5.2), we have (1.7).

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Riassunto

In questo lavoro si danno stime della rapidità di convergenza della soluzione di una disequazione variazionale di tipo parabolico verso la soluzione del problema omogeneizzato in dipendenza della regolarità dell'ostacolo e del dato iniziale.

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