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On Calkin's theorem (**)

T. Kato, in his treatment of perturbation theory ([4]), has introduced the concept of a strictly singular operator. A linear operator $A: E \to F$, where E and F are normed spaces, is *strictly singular* if, for every infinite dimensional closed subspace $M \subset E$ the restriction of A to M is not a linear homeomorphism.

If E and F are Banach spaces (as we will suppose throughout this paper) the strictly singular operators form a closed subspace $\mathscr{S}(E, F)$ of the space $\mathscr{B}(E, F)$ of all bounded linear operators. Moreover $\mathscr{S}(E, F)$ contains the closed subspace $\mathscr{K}(E, F)$ of all compact operators and when $E = F$, $\mathscr{S}(E) = \mathscr{S}(E, E)$ is a closed two-sided ideal of $\mathscr{B}(E, E) = \mathscr{B}(E)$.

Generally the conjugate $A': F' \to E'$ of a strictly singular operator A need not be strictly singular. To relate the strict singularity of A to that of A' and viceversa, R. J. Whitley ([8]) has introduced the following concepts.

A normed linear space E is *subprojective*, if given any closed infinite dimensional subspace M of E, there exists a closed infinite dimensional subspace N of M and a bounded projection from E onto N.

A normed linear space E is *superprojective* if, given any closed subspace M with infinite codimension, there exists a closed subspace N containing M, where N has infinite codimension, and a bounded projection from E onto N.

Let us denote by $\mathscr{M}(E, F)$ the set of all bounded linear operators $A: E \to F$ having the property that every closed subspace contained in the range $A(E)$ of A is finite dimensional. In [2] Calkin has shown that if H is a separable Hilbert space, the set $\mathscr{M}(H) = \mathscr{M}(H, H)$ is an ideal and coincides with

$$\mathscr{K}(H) = \mathscr{S}(H) = \text{the unique closed ideal of } \mathscr{B}(H) .$$

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In a non separable Hilbert space the equality $\mathscr{K}(H) = \mathscr{S}(H) = \mathscr{M}(H)$ is still true, although in this case $\mathscr{K}(H)$ is not the unique closed ideal. In [**3**] (see § 4.3) an example shows that generally, for an arbitrary Banach space E, we have $\mathscr{M}(E) \neq \mathscr{S}(E)$.

In this brief note we extend Calkin's theorem as follows.

Theorem. *Let E be reflexive, F any Banach space. If E is also superprojective together with all its closed subspaces, then $\mathscr{M}(E, F) = \mathscr{S}(E, F)$.*

Proof. Let $A \in \mathscr{M}(E, F)$, and N a closed subspace of E. If the restriction A_N of A on N is an homeomorphism, $A_N(N)$ is a closed subspace of the range $A(E)$, thus dimension of $A_N(N)$ = dimension of $N < +\infty$ i.e. A is a strictly singular operator.

Conversely let A be strictly singular and M a closed subspace of $A(E)$. Let

$$U = \{x \in E:\ Ax \in M\}$$

and denote by A_U the restriction of A on U. Since U is closed, directly from the definition of strict singularity, it follows that the operator $A_U: U \to M$ is still strictly singular. Moreover U is reflexive and by hypothesis is a superprojective Banach space. Then by corollary 4.7 of [**8**] we have that the dual space U' of U is a subprojective Banach space and by corollary 2.3 of [**8**] the conjugate of A_U, $A_U': M' \to U'$ is also strictly singular. Since $A_U(U) = M$ we have that A_U is a bounded surjective operator, hence its conjugate A_U' must be one-to-one. By the Open Mapping theorem it follows that A_U' has a bounded inverse, i.e. the operator A_U' is a linear homeomorphism of M' onto some subspace of U'. Then by the strict singularity of A_U' we must have $\dim M' < +\infty$ and from that $\dim M < +\infty$ i.e. $A \in \mathscr{M}(E, F)$.

Since any Hilbert space H is reflexive and superprojective together with all its closed subspaces, we have

Corollary I. *If H is a Hilbert space, F any Banach space, then $\mathscr{M}(H, F) = \mathscr{S}(H, F)$.*

A class of Banach spaces which are reflexive and superprojective together with any closed subspace is given by the class of *$\mathscr{L}_2$-spaces* treated in [**6**]. Such Banach spaces E are characterized by the fact that they are isomorphic to some Hilbert space, or equivalently (see [**6**], theorem 11.5.27) by the fact that they admit a bounded projection of E onto every closed subspace of E. Such spaces are subprojective, hence by Pfaffenberger's result [**7**] the ideal $\mathscr{S}(E)$ coincides with the ideal $\mathscr{I}(E)$ of the inessential operators (which generally contains properly $\mathscr{S}(E)$) introduced in [5] by D. Kleinecke. Then, by the theorem, we conclude that

Corollary II. *If the Banach space E is a $\mathscr{L}_2$-space, then $\mathscr{M}(E) = \mathscr{I}(E)$.*

Bibliography

[1] P. Aiena, *Relative regularity and Riesz operators*, Rend. Sem. Mat. Univ. Padova **67** (1982), 13-19.

[2] J. W. Calkin, *Abstract symmetric boundary conditions*, Trans. Amer. Math. Soc. **45** (1939), 369-442.

[3] I. C. Gohberg, A. S. Markus and I. A. Feld'man, *Normally solvable operators and ideals associated with them*, Trans. Amer. Math. Soc. (2) **61** (1967), 63-84.

[4] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. **6** (1958), 261-322.

[5] D. Kleinecke, *Almost finite, compact, and inessential operators*, Proc. Amer. Math. Soc. **14** (1963), 863-868.

[6] J. Lindestrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes **338**, Springer-Verlag 1973.

[7] W. E. Pfaffenberger, *On the ideals of strictly singular and inessential operators*, Proc. Amer. Math. Soc. **25** (1970), 603-607.

[8] R. J. Whitley, *Strictly singular operators and their conjugated*, Trans. Amer. Math. Soc. **113** (1964), 252-261.

Riassunto

Siano E ed F due spazi di Banach. Se E è inoltre riflessivo e superproiettivo assieme ad ogni suo sottospazio chiuso, lo spazio degli operatori strettamente singolari che vanno da E in F coincide con l'insieme degli operatori i cui codominii non possono contenere sottospazi chiusi di dimensione infinita. Ciò generalizza un teorema di Calkin valido per operatori che agiscono tra spazi di Hilbert.

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