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On von Neumann regular rings (IX) (**)

Introduction

Generalisations of quasi-injective and injective modules, noted IQC and MD-injective, are introduced to study von Neumann regular and associated rings. Left IQC rings are proved to be left continuous (in the sense of Utumi [**6**]) while left self-injective regular rings are characterised as left non-singular left IQC rings. If ${}_AM$ is either IQC or MD-injective whose complement left submodules are isomorphic to direct summands, then E/V is von Neumann regular, where $E = \text{End}\,({}_AM)$ and $V = \{f \in E/\ker f$ is essential in ${}_AM\}$ is the Jacobson radical of E. Semi-simple Artinian rings are characterised as rings whose left modules are MD-injective. A generalisation of von Neumann regular rings is also considered and several interesting properties are derived.

Throughout, A represents an associative ring with identity and A-modules are unitary. J, Z will denote respectively the Jacobson radical and the left singular ideal of A. A is called *left non-singular* (resp. *semi-simple*) iff $Z = 0$ (resp. $J = 0$). More generally, a left A-module M is called *non-singular* iff $Z(M)$, the left singular submodule, is zero.

An usual, (1) an ideal of A means a two-sided ideal; (2) a left (right) ideal of A is called *reduced* iff it contains no non-zero nilpotent element; (3) A is called a *left V-ring* iff every simple left A-module is injective [**3**]; (4) A left A-module M is called *p-injective* iff for any principal left ideal P of A, any left A-homorphism $g: P \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in P$. Then A is *von Neumann regular* iff every left A-module is p-injective. It is well-known that A is von Neumann regular iff every left A-module is flat. If I is a p-injective left ideal of A, then A/I is a flat left A-module $[\mathbf{7}]_4$.

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We here introduce the following definitions.

Def. 1. A left A-module M is called IQC, if, for any essential left submodule N such that there exists a non-zero complement left submodule of M isomorphic to a factor module of N, every left A-homomorphism of N into M may be extended to an endomorphism of ${}_AM$.

Def. 2. A left A-module M is called MD-*injective* if, for any left A-module P which is isomorphic to a direct summand of ${}_AM$ and any left A-monomorphisms f, g of P into M, there exists an endomorphism h of ${}_AM$ such that $hg = f$.

Obviously, any quasi-injective left A-module is IQC and any injective left A-module is MD-injective.

Left self-injective rings are generalised to left continuous rings by Utumi [**6**]. The notion of continuity has been extended to modules and studied by various authors (cfr. [**1**] and [**5**]). Recall that a left A-module M is continuous iff (a) any complement left submodule of M is a direct summand of ${}_AM$ and (b) any left submodule of M which is isomorphic to a direct summand of ${}_AM$ is a direct summand of ${}_AM$.

We proceed to prove that IQC left modules are intermediate between quasi-injective and continuous left modules (this justifies the notation).

Theorem 1. *If M is an* IQC *left A-module, then ${}_AM$ is continuous.*

Proof. We first prove that any non-zero complement left submodule C of M is a direct summand of ${}_AM$. Let K be a relative complement of ${}_AC$ in ${}_AM$ such that $E = C \oplus K$ is an essential left submodule of M. Suppose that $E \neq M$. If $p: E \to C$ is the canonical projection, then by Zorn's Lemma, the set of submodules N of ${}_AM$ containing E such that p extends to a left A-homomorphism from N into C has a maximal member Q. Let $h: {}_AQ \to {}_AC$ be the extension of p to Q. If $i: C \to M$ is the inclusion map, since ${}_AM$ is IQC and $Q/\ker h \approx C$, then $ih: {}_AQ \to {}_AM$ extends to an endomorphism g of ${}_AM$. Suppose that $g(M) \not\subseteq C$. Since C is a relative complement of K in ${}_AM$ ([**4**]$_1$, Proposition 1.4), then $(g(M) + C) \cap K \neq 0$. Let $0 \neq k \in K \cap (g(M) + C)$, $k = g(m) + c$, $m \in M$, $c \in C$. Then $L = \{z \in M / g(z) \in E\{$ is a submodule of ${}_AM$ which strictly contains Q (because $g(m) \notin C$ and hence $m \notin Q$ but $g(m) = k - c \in E$). If $r: L \to E$ is the map defined by $r(y) = g(y)$ for all $y \in L$, then $pr: L \to C$ is an extension of h to L and therefore an extension of p to L, which contradicts the maximality of Q. This proves that $g(M) \subseteq C$ whence $g(M) = C$ showing that whether $E = M$ or not, the epimorphism p extends to an epimorphism $g: M \to C$. Now since $\ker g \cap C = 0$ and for any $u \in M$,

$u = g(u) + (u - g(u))$, where $g(u) \in C$, $g(u - g(u)) = g(u) - g^2(u) = g(u) - g(u) = 0$ which yields $M = C \oplus \ker g$. Next, we prove that if D is a left submodule of M isomorphic to ${}_AC$, then ${}_AD$ is a direct summand of ${}_AM$. If I is a relative complement of ${}_AD$ in ${}_AM$, then $B = D \oplus I$ is an essential left submodule of M. If $v: C \to D$ is an isomorphism, $w: D \to C$ the inverse isomorphism, $s: B \to C$ the extension of w to B, the preceeding proof then shows that s may be extended to $t: {}_AM \to {}_AC$. If $j: D \to M$ is the inclusion map, $y = vt$: $M \to D$ and for any $d \in D$, $yj(d) = vtj(d) = vt(d) = vs(d) = vw(d) = d$ which shows that yj is the identity map on D. This proves that ${}_AD$ is a direct summand of ${}_AM$, whence M is a continuous left A-module.

A is called a *left* IQC *ring* iff ${}_AA$ is IQC.

Corollary 1.1. *Let A be a left* IQC *ring. If I is an ideal of A such that ${}_AI$ is non-singular, then I is a von Neumann regular ring. Consequently, any reduced ideal of A is a strongly regular ring* (cfr. ([**6**]), Lemma 4.1).)

If L is an essential left ideal of a left IQC ring A containing a non-zero idempotent, then any left A-homomorphism of L into A extends to an endomorphism of ${}_AA$. Since continuous regular rings need not be self-injective (even with non-zero socle) ([**6**], p. 172), the next corollary then shows that IQC left modules form a proper subset of continuous left modules.

([**6**], Lemma 4.1) and Corollary 1.1 yield the following nice characterisation of self-injective regular rings.

Corollary 1.2. *The following conditions are equivalent:*

(1) *A is left self-injective regular;*

(2) *A is a semi-simple left* IQC *ring;*

(3) *A is a left non-singular left* IQC *ring.*

Corollary 1.3. *A primitive ring is left self-injective regular iff it is left* IQC.

Left self-injective regular rings need not be left V-rings ([**3**], p. 107). Recall that A is ELT (resp. MELT) iff every essential (resp. maximal essential, if it exists) left ideal of A is an ideal.

([**7**]$_5$, Lemma 1.1) and Corollary 1.2 yield

Corollary 1.4. *A semi-prime* ELT *left* IQC *ring is a left and right self-injective regular left and right V-ring of bounded index.*

Applying ([**2**], Corollary 20.3E), ([**5**], Lemma 2.3) to Theorem 1, we get

Corollary 1.5. *If the direct sum of any two* IQC *left A-modules is* IQC, *then any* IQC *left A-module is injective. In that case, A is a left Noetherian left V-ring.*

It is well-known (O. E. Villamayor) that A is a left V-ring iff every left ideal of A is an intersection of maximal left ideals.

Corollary 1.6. *The following conditions are equivalent:*

(1) *A is a left self-injective regular left V-ring;*

(2) *A is a left* IQC *ring such that any proper left ideal which contains every minimal projective left ideal of A is an intersection of maximal left ideals.*

Proof. Apply ($[7]_1$, Proposition 3) and ($[7]_3$, Theorem 1) to Theorem 1.

We now characterise rings whose p-injective left modules are MD-injective.

Theorem 2. *The following conditions are equivalent:*

(1) *A is a left Noetherian ring whose p-injective left modules are injective;*

(2) *every p-injective left A-module is* MD-*injective.*

Proof. (1) implies (2) evidently.

Assume (2). Let M be a p-injective left A-module, H the injective hull of ${}_AM$. Write $Q = {}_AM \oplus {}_AH$ and $D =$ the set of ordered pairs $(y, 0)$ for all $y \in M$. Then ${}_AD$ is a direct summand of ${}_AQ$ and ${}_AM \approx {}_AD$. If $i: M \to H$ is the inclusion map, $j: M \to Q$ and $k: H \to Q$ the canonical injections, since ${}_AQ$ is p-injective, then it is MD-injective by hypothesis, which implies there exists a left A-homomorphism $g: Q \to Q$ such that $gki = j$. If $p: Q \to M$ is the canonical projection, then $u = pgk: {}_AH \to {}_AM$ such that $ui = pj =$ identity map on M. This proves that ${}_AM$ is a direct summand of ${}_AH$, whence $M = H$ is injective. Since any direct sum of p-injective left A-modules is p-injective, then (2) implies (1) by ([**2**], Theorem 20.1).

The next two results connect IQC and MD-injective modules.

Theorem 1 and the proof of Theorem 2 yield the following MD-injective analogue of ([**2**], Proposition 20.4B).

Theorem 3. *The following conditions are equivalent*:

(1) *any* MD-*injective left A-module is injective*;

(2) *the direct sum of any two* MD-*injective left A-modules is* MD-*injective*;

(3) *the direct sum of any two* MD-*injective left A-modules is* IQC.

([**2**], Theorem 24.20), ([**6**], Theorem 7.10), Theorem 1 and the proof of Theorem 2 also yield the next result.

Theorem 4. *The following conditions are equivalent*:

(1) *A is quasi-Frobeniusean*;

(2) *A is a left and right Artinian* IQC *ring*;

(3) *the direct sum of any injective and any projective left A-modules is* MD-*injective.*

An element a of A is called *left regular* iff $l(a) = 0$. Call A *a left* MD-*injective ring* if ${}_AA$ is MD-injective.

Proposition 5. *Let A be a left* MD-*injective ring. Then*

(1) *any left regular element of A is right invertible*;

(2) $Z \subseteq J$;

(3) *every left or right A-module is divisible.*

Proof. (1) If $c \in A$ such that $l(c) = 0$, $f: Ac \to A$ the left A-monomorphism defined by $f(ac) = a$ for all $a \in A$, $i: Ac \to A$ the inclusion map, since ${}_AAc \approx {}_AA$, there exists a left A-homomorphism $h: A \to A$ such that $hi = f$. *If* $h(1) = d$, $1 = f(c) = hi(c) = h(c) = ch(1) = cd$ which proves (1).

(2) If $z \in Z$, for any $a \in A$, $l(1 - za) = 0$ implies $(1 - za)v = 1$ for some $v \in A$. This proves that $z \in J$.

(3) If c is a non-zero-divisor of A, then $cd = 1$ for some $d \in A$ by (1). Now $cdc = c$ and $r(c) = 0$ imply $dc = 1$ which proves c invertible. Then for any left (resp. right) A-module M, $M = cM$ (resp. $M = Mc$).

Let us now turn to a class of rings with special cyclic modules which generalise von Neumann regular rings.

Write « A satisfies (*) » if, for any maximal right ideal R of A, any $b \in R$, there exists a positive integer n such that A/b^nR is a flat right A-module.

Note that a local ring A such that $J^2 = 0$ satisfies (*). Following [**2**], A is called a *lift/rad ring* if, for any $a \in A$ such that $a^2 - a \in J$, there exists an idempotent $e \in A$ such that $e - a \in J$.

Proposition 6. *Let A satisfy* (*). *Then*

(1) *any left regular element is right invertible*;

(2) $Z \subseteq J$;

(3) *every left or right A-module is divisible*;

(4) *if P is a reduced principal right ideal of A, then $P = eA$, where e is an idempotent such that $(1-e)A$ is an ideal of A*;

(5) *A is a lift/rad ring.*

Proof. (1) If $c \in A$ such that $l(c) = 0$, suppose that $cA \neq A$. If M is a maximal right ideal containing cA, there exists a positive integer n such that $A/c^n M_A$ is flat. This implies that for any left ideal I, $I \cap c^n M = c^n MI$. In particular, $c^{n+1} = c^n d c^{n+1}$ for some $d \in M$. Now $(1-c^n d)c^{n+1} = 0$ implies $c^n d = 1$ (because $l(c) = 0$), which contradicts $cA \neq A$. This proves (1).

(2) and (3) are proved as in Proposition 5.

(4) If $P = aA$ is a reduced principal right ideal, then $l(a) \subseteq r(a)$ and if $aA + r(a) \neq A$, let M be a maximal right ideal containing $aA + r(a)$. Then $A/a^n M_A$ is flat for some positive integer n, which implies $a^{n+1} = a^n u a^{n+1}$ for some $u \in M$. Now P reduced implies $l(a^{n+1}) \subseteq r(a^{n+1}) = r(a)$ and therefore $(1 - a^n u) \in l(a^{n+1}) \subseteq r(a) \subseteq M$ implies $1 \in M$, contradicting $M \neq A$. This proves that $aA + r(a) = A$ and we get $a = a^2 b$ for some $b \in A$. Then P reduced implies $a = aba$ and $P = eA$, where $e = ab$ is idempotent. Since $(ed - ede)^2 = 0$ for any $d \in A$ and P is reduced, then $eA(1-e) = 0$ implies $A(1-e) \subseteq r(e) = (1-e)A$ which proves that $(1-e)A$ is an ideal of A.

(5) Let $a \in J$. If $aA + r(a) = A$, then $a = a^2 b$, $b \in A$, and since $(1-ab)$ is right invertible, then $a(1-ab) = 0$ implies $a = 0$. Therefore if $a \neq 0$, let M be a maximal right ideal containing $aA + r(a)$. The proof of (4) then shows that there exists a positive integer n and $u \in M$ such that $(1 - a^n u)a^{n+1} = 0$. Since $(1 - a^n u)$ is left invertible, $a^{n+1} = 0$ which proves that J is a nilideal. Then (5) follows from ([**2**], Proposition 18.21).

Corollary 6.1. *A is strongly regular iff A is a reduced ring satisfying* (*).

We now mention two results analogous to a well-known theorem of C. Faith-Y. Utumi (cfr. $[\mathbf{4}]_1$, Theorem 2.16) concerning quasi-injective modules. In the next two results, M denotes a left A-module, $E = \mathrm{End}\,({}_A M)$, $V = \{f \in E/\ker f$ is essential in ${}_A M\}$.

Proposition 7. *Let M be an* IOC *left A-module. Then*

(1) *E/V is von Neumann regular and V is the Jacobson radical of E*;

(2) *E is a lift/rad ring.*

Proposition 8. *Let M be a* MD*-injective left A-module. Then V is an ideal of E which is contained in the Jacobson radical of E. If, in addition, every complement left submodule of M is isomorphic to a direct summand of ${}_AM$, then V is the Jacobson radical of E and E/V is von Neumann regular.*

We are now in a position to characterise semi-simple Artinian rings in terms of IQC, MD-injective modules and rings satisfying (*). If every divisible singular left A-module is injective, then A is left hereditary. If A is a semi-prime left Goldie ring, then it is well-known that every essential left ideal of A contains a non-zero-divisor. Then ([**2**], Theorem 24.20), ([**5**], Lemma 2.3), ([**7**]$_4$, Theorem 2.4), Theorem 3, Proposition 6 and the proof of Theorem 2 yield

Theorem 9. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*

(2) *every finitely generated left A-module is* IQC;

(3) *every left A-module is* MD*-injective;*

(4) *A is a* MELT *ring such that the direct sum of any two* IQC *left A-modules is* IQC;

(5) *A is a left* IQC *ring whose divisible singular left modules are injective;*

(6) *A is a semi-prime ring whose* MD*-injective left modules coincide with flat left modules;*

(7) *A is a semi-prime left* MD*-injective, left or right Goldie ring;*

(8) *A is a semi-prime left or right Goldie ring satisfying* (*).

We conclude with a few remarks.

Remark 1. Since a reduced left ideal is left non-singular, Proposition 6(4) ensures that ([**1**], Corollary 6) holds for rings satisfying (*).

Remark 2. If A is a prime ring satisfying (*), then (a) the centre of A is a field, (b) either A is a division ring or every non-zero ideal of A contains a non-zero nilpotent element. (This is motivated by ([**7**]$_2$, Remark).

Remark 3. A semi-prime left IQC ring with essential left socle is left self-injective regular. (Such rings need not satisfy the maximum condition on left annihilators).

Remark 4. Let A be a left IQC ring satisfying any one of the following conditions (1) A contains a non-zero non-singular left ideal or (2) A has non-zero p-injective left socle. Then A is left self-injective. It follows that a left IQC ring is either left self-injective or each of its non-zero left ideals contains a non-zero nilpotent element belonging to its left singular ideal.

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