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On systems of cogenerators of epireflective subcategories (**)

Let $\underline{C}$ be an epireflective subcategory of a topological (or an initially structured) category $\underline{A}$ and for any cardinal number α let $\underline{C}(\alpha)$ be the class of objects of $\underline{C}$ such that $X \in \underline{C}(\alpha)$ iff the cardinal number of the set UX is less than or equal to α. The aim of this paper is to prove that we can associate to $\underline{C}$ a class K of $\underline{A}$-objects such that for any cardinal number α, there exists an object Y_α belonging to K with the property that any object Y of $\underline{C}(\alpha)$ is Y_α-initial, in the sense that Y is initial with respect to $\big(U(\underline{A}(Y, Y_\alpha)), UY_\alpha\big)$. The class K will be called *system of cogenerators* for $\underline{C}$. Such a paper is precisely a generalization of Giuli's result in topological spaces [1]. Examples are provided.

1 - Epireflective subcategories of topological categories and systems of cogenerators

Let $\underline{A}$ be a topological category as defined by H. Herrlich $[2]_3$, i.e. a concrete category (A, U) which satisfies: (1) Existence of initial structures. (2) Fibre smallness. (3) Terminal separator property.

1.1 - Def. Let X and Y be two $\underline{A}$-objects. X is called *Y-initial* iff X is initial with respect to the data $\big(U(\underline{A}(X, Y)), UY\big)$.

For every $Y \in \underline{A}$ we denote by IN_Y the full and isomorphism-closed subcategory of $\underline{A}$ whose objects are Y-initial.

For any subcategory $\underline{C}$ of $\underline{A}$ we denote by $\underline{C}(\alpha)$ the class of $\underline{C}$-objects $\{X \in \underline{C}$ s.t. card $(UX) \leqslant \alpha$ (the cardinality of UX is less than or equal to $\alpha)\}$. Clearly $\underline{C}(\alpha) \subset \underline{C}(\alpha')$ if $\alpha \leqslant \alpha'$.

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1.2 – Def. A subcategory $\underline{C}$ of a category $\underline{B} \subset \underline{A}$ is called *weakly cogenerated* in $\underline{B}$ iff for every cardinal number α, there exists an $\underline{A}$-object A such that $\underline{C}(\alpha) = IN_A(\alpha) \cap \underline{B}$. The class $\{A_\alpha\}$ is called *a system of cogenerators* for $\underline{C}$. $\underline{C}$ is called *simply cogenerated* if the class of cogenerators is a set.

The following lemmas are easy to prove.

1.3 – Lemma. *Let T, X, Y be $\underline{A}$-objects. If T is X-initial and X is Y-initial, then T is Y-initial.*

1.4 – Lemma. *If ΠZ_j is the product of a family of $\underline{A}$-objects $\{Z_j\}$ in $\underline{A}$, then there exists an embedding $Z_j \xrightarrow{k_j} \Pi Z_j$, for each $j \in J$.*

As in [**4**], $\underline{A}_0$ will denote the largest epireflective subcategory of $\underline{A}$ which is not bireflective.

1.5 – Lemma. *Let X be a Y-initial $\underline{A}$-object. If X belongs to $\underline{A}_0$, then $U(\underline{A}(X, Y))$ separates the points.*

1.6 – Theorem. *Let $\underline{C}$ be a full and isomorphism-closed subcategory of $\underline{A}$. The following statements are equivalent:*

(a) *$\underline{C}$ is epireflective in $\underline{A}$;*

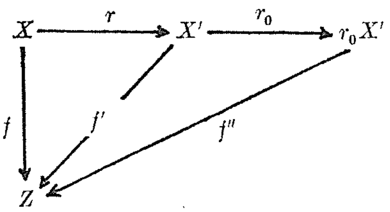
(b) *$\underline{C}$ is closed under the formation of products and extremal subobjects;*

(c) *$\underline{C}$ is weakly cogenerated.*

Proof. (a) $\Leftrightarrow$ (b). It follows from general results [**3**] since $\underline{A}$ is a co-(well powered) (epi, embedding)-category.

(b) $\Rightarrow$ (c). If $\underline{C} \not\subset \underline{A}_0$ ($\underline{C} \subset \underline{A}_0$), we have to prove that for each cardinal number α, there exists an $\underline{A}$-object A such that $\underline{C}(\alpha)=IN_A(\alpha)$ ($\underline{C}(\alpha)=IN_A(\alpha)\cap\underline{A}_0$). Let $\mathscr{B}(\alpha)$ be a family of objects of $\underline{C}(\alpha)$ pairwise non isomorphic such that for each $X \in \underline{C}(\alpha)$ there exists $X' \in \mathscr{B}(\alpha)$ with $X \simeq X'$, except the empty set. $\mathscr{B}(\alpha)$ is a set obviously. Let A be the product of all objects of $\mathscr{B}(\alpha)$. From **1.3** and **1.4** and by the fact that every $X \in \underline{C}(\alpha)$ is isomorphic to a factor of A, it follows that X is A-initial. Thus $\underline{C}(\alpha) \subset IN_A(\alpha)$ ($\underline{C}(\alpha) \subset IN_A(\alpha) \cap \underline{A}_0$). Conversely, if $\underline{C} \not\subset \underline{A}_0$, then $\underline{C} \supset IND$ [**4**], so A is a coseparator [**2**]$_3$. Hence the total source from X to A, $\underline{A}(X, A)$, is a monosource [**3**]. Since X is A-initial, $\underline{A}(X, A)$ is an extremal monosource. Thus X is an extremal subobject of A^I and, by (b) $X \in C(\alpha)$. If $\underline{C} \subset \underline{A}_0$, every $X \in IN_A(\alpha) \cap \underline{A}_0$ is initial with respect to the source $U(\underline{A}(X, A))$. By **1.5** such a source is a monosource. *So* X is an extremal subobject of A^I. Thus $IN_A(\alpha) \cap \underline{A}_0 \subset \underline{C}(\alpha)$.

(c) $\Rightarrow$ (a). If $\underline{C} \not\subset \underline{A}_0$, let X be an $\underline{A}$-object with card $(UX)=\alpha$ and let X' be the initial lift of $\overline{(U(\underline{A}(X, A)), UA)} \cdot X' \in \underline{C}(\alpha) \subset \underline{C}$. Let $r: X \to X'$ be the $\underline{A}$-morphism such that $Ur = I_{UX}: UX \to UX' = UX$. Now let Z be a $\underline{C}$-object and $f: X \to Z$ an $\underline{A}$-morphism. We suppose card $(UZ)=\alpha'$. Since $\underline{C}$ is weakly cogenerated, then Z is initial with respect to $U(\underline{A}(Z, A'))$, where A' is the cogenerator relative to the cardinal number α'. If we take $\alpha'' = \max(\alpha, \alpha')$, then X and Z are both A''-initial. Let X and Z be A''-initial with respect to $Ug_h = U(\underline{A}(X, A''))$ and $Ug_k = U(\underline{A}(Z, A''))$. Since $Ug_k Uf = U(g_k \circ f)$, by the initiality there exists a unique $f': X' \to Z$ such that $Uf' = Uf \circ I_{UX}^{-1}$ (I_{UX}^{-1} is the inverse of I_{UX}) and $f = f' \circ r$. If $\underline{C} \subset \underline{A}_0$ let $X \in \underline{A}$, $r: X \to X'$ be as above and let $r_0: X' \to r_0 X'$ be the $\underline{A}_0$-reflection of X'. The following commuting diagram



shows that for every f from X to $Z \in \underline{C}$, there exists f'' from $r_0 X'$ to Z such that $f'' \cdot r_0 \cdot r = f$.

Examples

(1) TOP$_0$ is simply cogenerated by the Sierpinski space S [**1**].

(2) TYCH is simply cogenerated by the real line R [**1**].

(3) TOP$_1$ is a concrete example of a weakly cogenerated subcategory of TOP. In fact TOP$_1$ is not simply cogenerated. If we take for any cardinal number α the topological space A whose underlying set has cardinality α, endowed with cofinite topology, we obtain a class of cogenerators for TOP$_1$ [**1**].

(4) Let UNIF be the topological category of all uniform spaces. The subcategory UNIF$_{LF}$ of all uniform spaces with a point-finite base is epireflective in UNIF. It is weakly cogenerated by the class $\{C_0(\alpha)\}$, (see [**6**], corollary 2.5).

2 - Initially structured categories and systems of cogenerators

We know that categories whose objects are topological spaces satisfying a separation axiom usually do not form a topological category in Herrlich's sense. In order to extend Theorem **1.6** to these categories and to many others, we are going to prove it in the initially structured categories as defined by L. D. Nel [**5**]. Such categories are wider than topological categories and include known categories such as Hausdorff spaces which are not topological.

2.1 – Def. A category A is *initially structured with forgetful functor* U if there exists a functor $U: A \to \text{SET}$ such that:

(1) Any U-source $(X \xrightarrow{f_i} UA_i)$ in SET has an (epi, monosource)-factorization $(X \xrightarrow{e} UB \xrightarrow{Ug_i} UA_i)$ with $(B \xrightarrow{g_i} A_i)$ an initial source.

(2) U has small fibres, i.e., for every object X in SET there is at most a set of pairwise non isomorphic A-objects A with $UA = X$.

(3) There is precisely one object P (up to isomorphism) such that UP is terminal and separating in SET.

Since (1) is equivalent to say that U is (epi, monosource)-topological in the sense of $[\mathbf{2}]_4$, many properties of $\underline{A}$ and U are known from $[\mathbf{2}]_4$.

As in the case of topological categories, we have

2.2 – Lemma. *If* $\{Z_j\}$ *is a family of* $\underline{A}$*-objects and* ΠZ_j *is their product, then there exists an embedding* $k_j: Z_j \to \Pi Z_j$, *for all* j.

If $\underline{C}$ is a subcategory of $\underline{A}$, $\underline{C}(\alpha)$ denotes the class of all $\underline{C}$-objects X such that card (UX) is less than or equal to α.

Let (E, M) be one of the following pairs: (epi_U, extremal monosources) or (extremal epi, monosources).

2.3 – Lemma. *For every* M*-source* $(a_i: A \to A_i)_I$ *in* $\underline{A}$, *there exists a set* $J \subset I$ *such that* $(a_i: A \to A_i)_J$ *is an* M*-source.*

Let $\{Y^i\}$ be a family of $\underline{A}$-objects. M_{Y^i} denotes the class of all $X \in \underline{A}$ such that there exists an M-source from X to $\{Y^i\}$.

2.4 – Def. A subcategory $\underline{C}$ of the category $\underline{A}$ is called *weakly* M*-cogenerated* iff for any cardinal number α, there exists an $\underline{A}$-object A such that $\underline{C}(\alpha) = M_A(\alpha)$. A is called *the cogenerator* for the C-obiects with cardinal number less than or equal to α.

2.5 – Theorem. *Let* $\underline{C}$ *be a subcategory of an initially structured category* $(\underline{A}, U)$. *The following are equivalent*:

(a) $\underline{C}$ *is* E*-reflective in* $\underline{A}$;
(b) $\underline{C}$ *is closed under the formation of products and* M*-subobjects*;
(c) $\underline{C}$ *is weakly* M*-cogenerated*;
(d) $\underline{C}$ *is closed with respect to* M*-sources.*

Proof. (a) $\Leftrightarrow$ (b). It follows from the fact that $\underline{A}$ is an E-co-(well powered) (E, M)-category [3].

(b) $\Rightarrow$ (c). Let $X \in \underline{C}(\alpha)$ and let A be given as in (b) $\Rightarrow$ (c) of Theorem **1.6**. Since card $(UX) \leqslant \alpha$, then X is a factor of the product object A and since from Lemma **2.2** it is an extremal subobject of A, it belongs to $M_A(\alpha)$. Now let $X \in M_A(\alpha)$. Since there exists an M-source $(f_i\colon X \to A)_I$, then $\Pi f_i\colon X \to A^I$ is an M-morphism. Thus $X \in \underline{C}(\alpha)$.

(c) $\Rightarrow$ (d). $\underline{C}(\alpha) = M_A(\alpha)$, for every α. Let $m_i\colon X \to B_i$ be an M-source with $B_i \in \underline{C}$. Since for every B_i there exists an M-source $f_j^i\colon B_i \to A_i$, then $\{f_j^i \circ m_i\}_{ij}$ is an M-source. Note that we can consider $\{f_j^i \circ m_i\}_{ij}$ as a set of $\underline{A}$-morphisms by Lemma **2.3**. So let $\alpha' = \text{card}\,(\{\bigcup_i UA_i\})$ and A' the cogenerator for $\underline{C}(\alpha')$. Since A_i, for all i, belongs to $M_{A'}(\alpha')$, then there exists an M-source $n_k^{ij}\colon A^i \to A'$ and $\{n_k^{ij} \circ f_j^i \circ m_i\}_{ijk}$ is an M-source. Thus $X \in M_{A'}(\alpha') = \underline{C}(\alpha') \subset \underline{C}$.

(d) $\Rightarrow$ (b). Let $\{B_i\} \subset ob\underline{C}$ and ΠB_i their product in $\underline{A}$. Since $\Pi_i\colon \Pi B_i \to B_i$ is an M-source, then $\Pi B_i \in \underline{C}$. Since $\underline{C}$ is closed with respect to M-sources, in particular it is closed with respect to M-morphisms.

References

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Summary

The aim of this paper is to give a characterization of epireflective subcategories of a topological (or an initially structured) category A, in terms of initial structures. In other words, we give a way to associate a system of cogenerators to any epireflective subcategory of A.

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