Riv. Mat. Univ. Parma (4) 9 (1983), 289-294

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A quasi-analytical treatment of a singular integral equation (**)

1 - Introduction

In the present paper the following Fredholm integral equation of second kind, related to a large class of physical problems described by a linearized model of the Boltzmann equation [4] is studied

$$f(x) = g(x) + Af(x) , \tag{1}$$

where A is a singular Fredholm operator, $f(x)$ is the unknown function and $g(x)$ is a known function which characterizes the particular physical problem considered.

The difficulty of solving problem (1) is due to the fact that the kernel of the integral operator cannot be analytically expressed and is singular in the origin; however such an equation has been studied by some authors (see, in particular, the paper by Boffi et al. [3] and its related bibliography, where this equation is applied to internal flows between parallel plates in rarefied gas conditions).

In the present paper a new method, which is devoted to the approximation of the afore-mentioned kernel, is developed; moreover the evaluation of the error-bounds of the approximated solution of Eq. (1) is realized. In particular **2** describes the approximation method, whereas **3** performs some numerical results and applications.

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(**) Ricevuto: 12-II-1982.

2 - Analysis of the method

Let us consider the following equation

$$(2) \qquad f(x) = g(x) + Af(x) \equiv g(x) + \int_{-a}^{a} T(x,s) f(s)\, ds,$$

$x \in [-a, a]$, $A: L_2(-a, a) \to L_2(-a, a)$, with

$$(3) \qquad T(x, s) \overset{\Delta}{=} \pi^{-1/2} \int_0^\infty \frac{1}{t} \exp\{-t^2 - \frac{|x-s|}{t}\}\, dt \overset{\Delta}{=} \pi^{-1/2} T_{-1}(|x-s|),$$

where T_{-1} is the -1-order Abramowitz function [**2**].

Before developing the method, let us recall some important properties of the function T_{-1}, which will be used furtherly; in particular, putting

$$p = |x - s| \in P = [0, 2a],$$

we have (see refs. [**2**], [**3**])

$$(1) \qquad p \to +\infty: \quad T_{-1}(p) \simeq \frac{\pi^{1/3}}{3} \left(\frac{p}{2}\right)^{-1/3} \exp\{-3\left(\frac{p}{2}\right)^{2/3}\},$$

$$(2) \qquad p \to 0^+: \quad T_{-1}(p) = -\frac{3}{2}\gamma + \pi^{1/2} p - \ln p\,[1 + O(p^2)],$$

where $\gamma = 0.577\,215\,7$ is the Euler constant,

$$(3) \qquad \lim_{p \to 0^+} T_{-1}(p) = +\infty, \qquad (4) \qquad \|T_{-1}\| \leqslant c < +\infty.$$

Let us now define the following function in $L_2(P)$

$$B(p) = \pi^{1/2} p \exp\{p\} T(p),$$

the function B has the following properties

$$\text{(i)} \quad B(0) = 0, \qquad \text{(ii)} \quad \lim_{p \to +\infty} B(p) = +\infty, \qquad \text{(iii)} \quad \|B\| \leqslant k < +\infty,$$

which follow immediately from the properties of T_{-1}.

The regularity of B allows its approximation with a polynomial expansion Q_n (see ref. [**7**], pag. 36), such that

$$\lim_{n\to\infty} Q_n(p) = B(p) ,$$

moreover let us define the following functions

$$\forall n \geqslant 1 \qquad T_n(p) \in L_2(P), \qquad T_n(p) = \frac{Q_n(p)}{p \exp\{p\}} , \tag{5}$$

which, from the definition of B, has the folloving properties

(iv) $\lim_{p\to 0^+} T_n(p) = \beta < +\infty$, (v) $\|T_n\| \leqslant k' < +\infty$.

Finally let us define the following equation

$$f_n(x) = g(x) + A_n f_n(x) \equiv g(x) + \int_{-a}^{a} T_n(x, s) f_n(s)\, ds . \tag{6}$$

The aim of the present paper consists in studying eq. (6) instead of eq. (2) and in evaluating the error which is made following this line. For this purpose let us re-write eqs. (2) and (6), respectively, as follows

$$Hf = g , \tag{7}$$ $$H_n f_n = g, \tag{8}$$

where

$$H, H_n: L_2(-a, a) \to L_2(-a, a) , \quad H = I - A , \quad H_n = I - A_n \tag{9}$$

I being the identity operator.

Proposition 1. *The problem* (8) *is well-posed in the sense that exist sufficient conditions for the existence and uniqueness of the solution of eq.* (8).

Proof. Let us prove the existence of $(H_n)^{-1}$; in fact T_n is symmetric and bounded (see eq. (5) and property (v)), therefore A_n is a compact and self-adjoint operator (see ref. [**6**], pp. 456-459). These properties are sufficient to prove the existence and the uniqueness of the solution of eq. (8) (see again ref. [**6**], third theorem of Fredholm, sec. IX-2.4), as well as eq. (7) (see ref. [**3**]).

Therefore $(H_n)^{-1}$ exists and is bounded (see ref. [**5**]), namely

$$\|(H_n)^{-1}\| \leqslant w < +\infty \tag{10}$$

and the proposition holds.

Proposition 2. *If*

$$\|H - H_n\| \cdot \|(H_n)^{-1}\| < 1 \tag{11}$$

and

$$\|T - T_n\| \leqslant \varepsilon , \tag{12}$$

then the error in solving eq. (8) *instead of eq.* (7) *is controlled as follows*

$$\|f - f_n\| \leqslant \|(H)^{-1} - (H_n)^{-1}\| \cdot \|g\| \leqslant \alpha \|g\| , \tag{13}$$

with $\alpha = \alpha(\varepsilon, n)$.

Proof. By means of condition (10) together with hypothesis (11), it is possible to state (see ref. [**5**]) the following inequality

$$\|(H)^{-1} - (H_n)^{-1}\| \leqslant \frac{\|(H_n)^{-1}\|^2 \cdot \|H - H_n\|}{1 - \|(H_n)^{-1}\| \cdot \|H - H_n\|} . \tag{14}$$

On the other hand from the Fredholm inequality on compact operators (see ref. [**6**], sec. IX-2.1) the following inequalities are verified

$$\|H - H_n\| = \|A - A_n\| \leqslant \|T - T_n\| . \tag{15}$$

Consequently, utilizing the inequalities (14-15), the expression (13) is satisfied and

$$\alpha = \alpha(\varepsilon, n) = \frac{\|(H_n)^{-1}\|^2 \varepsilon}{1 - \|(H_n)^{-1}\| \varepsilon} , \tag{16}$$

which proves the proposition.

Then from formula (13) the distance $\|f - f_n\|$ can be sufficiently little because in the actual physical problems is $\|g\| < 1$.

3 - Applications and comments

In order to verify the conditions stated in Proposition 2, this section is devoted to an evaluation of the quantity ε which appears in the inequality (12). We prove that

$$\|T - T_n\| < (\varepsilon' + \varepsilon'')^{1/2}, \qquad \varepsilon', \varepsilon'' \in \boldsymbol{R}^+ . \tag{17}$$

For this purpose let us define G' and G''

$$\begin{aligned} G' &\overset{\Delta}{=} \|T - T_n\|^2, \qquad 0 < p < h \ll 2a, \\ G'' &\overset{\Delta}{=} \|T - T_n\|^2, \qquad h \leqslant p \leqslant 2a . \end{aligned} \tag{18}$$

The problem consists in proving that

$$G' \leqslant \varepsilon', \qquad G'' \leqslant \varepsilon'' .$$

The quantity ε'' can be evaluated numerically and it can be shown that ε'' is arbitrarly small increasing the degree n of the polynome Q_n (see eq. (4)). On the other hand, for evaluating ε'', it is necessary to utilize the properties of the Abramowitz function T_{-1}, because the functional G' contains the singularity in the point $p = 0$.

Proposition 3. *The functional G' is bounded and results*

$$\varepsilon' = O(h \cdot \ln^2 h) \qquad \textit{when } h \to 0^+ .$$

Proof. In fact $\forall p > 0$: $T_n(p) > 0$ and therefore results

$$G' \overset{\Delta}{=} \int_0^h [T(p) - T_n(p)]^2 \, \mathrm{d}p < \int_0^h [T(p)]^2 \, \mathrm{d}p ; \tag{19}$$

on the other hand it is known (see ref. [**3**]) that

$$T(p) \leqslant \tfrac{1}{2} + \pi^{-1/2} E_1(p) , \tag{20}$$

where $E_1(p)$ is the exponential integral function [**2**] defined as follows

$$E_1(p) \overset{\Delta}{=} -\gamma - \ln p + \sum_{m=0}^{\infty} c_m p^m \qquad (p > 0) . \tag{21}$$

Therefore from eq. (21) follows

$$G' < \int_0^h (T(p))^2\,dp = \int_0^h [\tfrac{1}{2} + \pi^{-1/2}(-\gamma - \ln p + \sum_{m=0}^{\infty} c_m p^m)]^2\,dp\,. \tag{22}$$

The analytical treatment of the last integral shows immediately that, when $h \to 0^+$, the quantity ε' is of the order of $(h \cdot \ln^2 h)$, which proves the proposition.

As a comment let us note that the last proposition shows that the objective of reducing ε' and therefore ε corresponds to the numerical problem of computing the function T_{-1} for the smallest value $p = h$.

Let us now deal with the quantity ε''. The latter can be evaluated with numerical calculations, once the function $B(p)$ has been approximated with a polynome $Q_n(p)$, $p \in P$.

For this purpose let us use a Bernstein polynome [7], namely

$$Q_n(x) = x \sum_{k=1}^{n} \binom{n}{k} B(k/n) x^{k-1}(1-x)^{n-k}, \quad x = \frac{p}{2a}\,.$$

Then ε'' decreases quickly increasing the order n of the polinome.

As an application for $n = 8$, ε'' is of the order of 10^{-9}, $\|(H_8)^{-1}\|$ of the order of 1.

Let us note that by this method the numerical computing is reduced; in fact the kernel is computed once for every value of the parameter a and this opens a way to the analytical treatment.

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