Riv. Mat. Univ. Parma (4) 9 (1983), 275-280

MURSAILEEN and H. Z. KHAN (*)

Nonarchimedean spaces of bounded sequences all of whose invariant means are equal (**)

1 - Introduction

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional Φ on l_∞ is said to be an invariant mean or a σ-mean if and only if: (1) $\Phi(x) \geqslant 0$ when the sequence $x = \{x_n\}$ has $x_n \geqslant 0$ for all n; (2) $\Phi(e)=1$, where $e = \{1, 1, 1, ...\}$; (3) $\Phi(\{x_{\sigma(n)}\}) = \Phi(x)$ for all $x \in l_\infty$. For certain kinds of mappings σ, every invariant mean Φ extends the limit functional on c in the sense that $\Phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal. In case σ is the translation mapping $n \to n+1$, a σ-mean is often called *a Banach limit* and V_σ is the set of f of almost convergent sequences [1].

It can be shown that the set V_σ can be characterized as

$$V_\sigma = \{x \in S\colon \lim_{m\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n \text{ exists uniformly in } n\},$$

and has the form Le, L being the common value of all σ-means at x (see [4]) where $Tx = \{x_{\sigma(n)}\}$. We write $L = \sigma - \lim x$. Also

$$V_\sigma(E) = \{x \in E\colon \lim_{m\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n \text{ exists uniformly in } n\}.$$

(*) Indirizzo: Dept. of Math. Aligarh Muslim University, Aligarh, 202001 India.
(**) Ricevuto: 12-I-1982.

Let $p = \{p_m\}$ be a sequence of real numbers such that $p_m > 0$ and $\sup p_m < \infty$. We define

$$V_\sigma(p) = \{x \in S: \lim_{m\to\infty} | \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n - Le|^{p_m} = 0 \quad \text{uniformly in } n \text{ for}$$

$$\text{some } L \in S\} ,$$

$$V_{0\sigma}(p) = \{x \in V_\sigma(p) \colon \sigma - \lim x = 0\} .$$

In particular, if $p_m = p$ for every m, $V_\sigma(p)$ and $V_{0\sigma}(p)$ are same as V_σ and $V_{0\sigma}$ respectively.

In [**6**] and [**3**]$_{1,2,3}$, some matrix transformations have been characterized in V_σ for real or complex sequences. In [**5**], authors study some matrix transformations in nonarchimedean spaces. In the present paper, we study the space V_σ as nonarchimedean and characterize some classes of matrices.

2 - Preliminaires

Let X denote a topological vector space over S. A subset Y of X is said to be *absolutely convex* if $VY + VY \subset Y$, V denotes the valuation ring of S. Y is said to be *convex* if it is absolutely convex or a translate of an absolute convex set. $Y \subset X$ is called *balanced* if $VY \subset Y$. A filter base consisting of convex sets is called a *convex filter base*. X is said to be *non-archimedean* if there exists a convex filter base of $0 \in X$ consisting of balanced sets.

Let $A = (a_{nk})$ be an infinite matrix of elements of S and $x = \{x_n\}$ be a sequence of elements of S. We write $Ax = \{A_n(x)\}$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n. If P and Q are any two sequence spaces, we write (P, Q) to denote the class of matrices A such that $Ax \in Q$ for every $x \in P$.

We write $\sup_n := \sup_{n=0,1,\ldots}$; $\lim_n := \lim_{n\to\infty}$; $\sum_n = \sum_{n=0}^{\infty}$.

Throughout this paper we shall use the notation $a(n, k)$ to denote the element a_{nk} of the matrix A. We write

$$t_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n , \quad t_{mn}(Ax) = \sum_{k=1}^{\infty} \sum_{j=0}^{m} a(\sigma^j(n), k)\, x_k/(m+1) ,$$

3 - Main results

Theorem 1. *If S is complete and $\inf p_m > 0$, then $V_\sigma(p)$ is a complete nonarchimedean topological vector space the nonarchimedean metric*

$$g(x) = \sup_{m,n} |t_{mn}(x)|^{p_m/M},$$

where $M = \max(1, \sup p_m)$. If E is complete then $V_\sigma(E)$ is complete.

Proof. Let $x, y \in V_\sigma(p)$. It is easy to show that $g(0) = 0$ and $g(x) = g(-x)$ and $g(x+y) \leqslant \max(g(x), g(y))$. Therefore g defines a nonarchimedean metric on $V_\sigma(p)$. Further for $\lambda \in S$ we have $g(\lambda x) \leqslant \max(1, |\lambda|)\, g(x)$. Therefore $\lambda \to 0$, $x \to 0 \Rightarrow \lambda x \to 0$ and if λ is fixed, $x \to 0 \Rightarrow \lambda x \to 0$. If $x \in V_\sigma(p)$ is fixed, then we have

$$g(\lambda x) \leqslant \max(|\lambda|, |\lambda|^{p'})\, g(x), \qquad \text{where } p' = \inf p_m .$$

Since $p_m/M \leqslant 1$, we have for every m and n (see [**2**]$_2$)

$$|t_{m,n}(x+y)|^{p_m/M} \leqslant |t_{mn}(x)|^{p_m/M} + |t_{mn}(y)|^{p_m/M},$$

and for every $\lambda \in S$ (see [**2**]$_1$) $|\lambda|^{p_m/M} \leqslant \max(1, |\lambda|)$.

Therefore it follows that $V_\sigma(p)$ is a linear topological space.

Now, let $Y_n(0) = \{x : g(x) < 1/n\}$, which is a filter base and each $Y_n(0)$ is balanced, for if $\lambda \in V$ and $x \in Y_n(0)$, then

$$g(\lambda x) \leqslant \max(1, |\lambda|)\, g(x) = g(x) < \frac{1}{n},$$

and therefore $x \in Y_n(0)$, i.e. $VY_n(0) \subset Y_n(0)$. Furthermore, each $Y_n(0)$ is convex, for if, $x, y \in Y_n(0)$ and $\lambda, \mu \in V$, then

$$g(\lambda x + \mu y) \leqslant \max(g(x)\, g(y)) < \frac{1}{n},$$

and hence $\lambda x + \mu y \in Y_n(0)$, i.e. $VY_n(0) + VY_n(0) \subset Y_n(0)$. This leads that $V_\sigma(p)$ is a nonarchimedean topological vector space.

To show completeness, let $\{x^i\}$ be a Cauchy sequence in $V_\sigma(p)$. Then for each k, $\{x_k^i\}$ is a Cauchy sequence in S and hence $x_k^i \to x_k$ for each k. Let

$x=\{x_k\}$, then $x^i \to x$. We now show that $x \in V_\sigma(p)$. Since $x^i \in V_\sigma(p)$, there exist $L^i \in S$ and m_i such that for every n and for every $m > m_i$

$$(1) \qquad |t_{m,n}(x^i - L^i e)|^{p_m/M} < \varepsilon .$$

Since $\{x_i\}$ is a Cauchy sequence, given $\varepsilon > 0$, there exists N_0 such that for $i, j > N_0$ and for every m, n,

$$(2) \qquad |t_{mn}(x^i - x^j)|^{p_m/M} < \varepsilon .$$

Taking limit as $j \to \infty$ we get

$$(3) \qquad |t_{mn}(x^i - x)|^{p_m/M} < \varepsilon .$$

By virtue of (1) and (2), it follows that, for $m > m_0$ and $i, j > N_0$, $|L^i e - L^j e|^{p_m/M} < \varepsilon$. Thus $\{L^i\}$ is a Cauchy sequence in S and therefore, there exists $L \in S$ such that, for $i > N_0$

$$(4) \qquad |L^i e - Le| < \varepsilon .$$

Now, by virtue of (1), (3) and (4), we have, for every $m > m_0$, $|t_{mn}(x) - Le|^{p_m/M} < \varepsilon$. This terminates the proof.

Theorem 2. $A \in (c(p), V_\sigma)$ *if and only if*: (i) *there exists an integer* $B > 1$ *such that* $\sup_{n,k,m} |\sum_{j=0}^{m} a(\sigma^j(n), k)| B^{-1/p_k}/(m+1) < \infty$, (ii) $a_{(k)} = \{a_{nk}\}_{n=i}^{\infty} \in V_\sigma$ *for each* k, (iii) $a = \{\sum_k a_{nk}\}_{n=i}^{\infty} \in V_\sigma$.

In this case, the σ-limit of Ax is $(\lim x)[u - \sum_k u_k] + \sum_k u_k x_k$ for every $x \in V_\sigma(p)$, where $u = \sigma - \lim a$ and $u_k = \sigma - \lim a_{(k)}$.

Proof. Let $A \in (c(p), V_\sigma)$. Since e^k, $e \in c(p)$, necessity of (ii) and (iii) is obvious, where $e^k = \{0, 0, \ldots, 0, 1\ (k\text{-th place}), 0, \ldots\}$. It is easy to see that $(c(p), V_\sigma) \subset (c_0(p), V_\sigma)$, therefore, for the necessity of (i) we observe that $A \in (c_0(p), V_\sigma)$, where $A \in (c(p), V_\sigma)$. It is obvious that $\{t_{mn}(Ax)\}$ is a sequence of continuous linear functionals on $c_0(p)$ such that $\lim_m t_{mn}(Ax)$ exists uniformly in n. Then, by uniform boundedness principle, there exists a sphere $S[\theta, \delta] \subset c_0(p)$, with $0 < \delta < 1$, $\theta = \{0, 0, 0, \ldots\}$, and a constant K such that $t_{mn}(Ax) \leqslant K$ for each m and for every $x \in S[\theta, \delta]$. For every integer $r > 0$, we define a sequence $(x^{(r)})$ of elements of $c_0(p)$ as follows

$$x_k^{(r)} = \begin{cases} \delta^{M/p_k} \ \operatorname{sgn}\ \left(\sum_{j=0}^{m} a(\sigma^j(n), k)/(m+1)\right), & 0 \leqslant k \leqslant r \\ 0, & r < k . \end{cases}$$

Then $x^{(r)} \in S[\theta, \delta]$ for every r and $\sup_k |\sum_{j=0}^{m} a(\sigma^j(n), k)| B^{-1/p_k}/(m+1) \leqslant K$, for each m and r, where $B = \delta^{-M}$. Therefore (i) is satisfied.

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied and $x \in c(p)$. Then there exists l, such that $|x_k - l|^{p_k} \to 0$. It is easy to check that $(u_k) \in c_0(p)$. Given $\varepsilon > 0$ there exists k_0 such that

$$|x_k - l|^{p_k/M} \leqslant \frac{\varepsilon}{B(2D+1)} < 1 \quad \text{for every } k > k_0,$$

where

$$D = \sup_{n,k,m} |t(n, k, m)| B^{-1/p_k}, \quad t(n, k, m) = \sum_{j=0}^{m} a(\sigma^j(n), k)/(m+1).$$

Therefore, we have

$$B^{1/p_k}|x_k - l| < B^{M/p_k}|x_k - l| < \left(\frac{\varepsilon}{2D+1}\right)^{M/p_k} < \frac{\varepsilon}{2D+1} \text{ for every } k > k_0,$$

where $M = \max(1, \sup p_k)$. By (i) and (ii) we have $|t(n, k, m) - u_k| B^{-1/p_k} < 2D$. Whence

$$|\sum_k (t(n, k, m) - u_k)(x_k - l)| \leqslant \sup_k |t(n, k, m) - u_k| \, |x_k - l| + \varepsilon,$$

$$\lim_m |\sum_k (t(n, k, m) - u_k)(x_k - l)| = 0 \qquad \text{uniformly in } n.$$

Therefore, $\lim_m \sum_k t(n, k, m) x_k = u + \sum_k u_k(x_k - l)$. This completes the proof.

References

[1] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167-190.

[2] I. J. Maddox: $[\cdot]_1$ *Spaces of strongly summable sequences*, Quart. J. Math. (18) **2** (1967), 345-355; $[\cdot]_2$ *Elements of functional analysis* (Camb. Univ. Press, 1970).

[3] Mursaleen: $[\cdot]_1$ *On infinite matrices and invariant means*, Indian J. Pure Appl. Math. (4) **10** (1979), 457-460; $[\cdot]_2$ *Invariant means and some matrix transformations*, Tamkang J. Math. (2) **10** (1979), 183-188; $[\cdot]_3$ *On some new invariant matrix methods of summability*, Quart. J. Math., Oxford (2) **34** (1983), 77-86.

[4] R. A. RAIMI, *Invariant means and invariant matrix methods of summability*, Duke Math. J. **30** (1963), 81-94.

[5] M. S. RANGACHARI and V. K. SRINIVASAN, *Matrix transformations in nonarchimedean fields*, Indag. Math. **26** (1964), 423-429.

[6] P. SCHAEFER, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. (1) **36** (1972), 104-110.

Abstract

Let l_∞ and c be the spaces of all bounded and convergent sequences of a nontrivially nonarchimedean valued field S and E be a nonarchimedean normed linear space over S. In this paper, we study some matrix transformations in nonarchimedean spaces.

* * *