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A generalization of the Borel theorem in diophantine approximation (**)

Let α be a real number, $[a_0; a_1, a_2, ...]$ be its simple continued fraction expansion, p_n/q_n be its n-th convergent. Let $|\alpha - p_n/q_n| = 1/(M_n q_n^2)$. E. Borel [1] proved that at least one of the three consecutive M's exceeds $\sqrt{5}$. A. Brauer and N. Macon [2] proved that either two of the five consecutive M's exceed $\sqrt{5}$ or at least one M exceeds 3. I can not find any information in the literature about the occasion for four consecutive M's. The purpose of this paper is to bridge this gap. We first prove that either two of the four consecutive M's exceed $\sqrt{5}$, or the sum of two M's exceeds $2\sqrt{5}$, then we give a slight improvement of Brauer and Macon's theorem.

We introduce some notations. Let $P=[a_{n+2}; a_{n+3}, ...]$, $Q=[a_{n-1}; a_{n-2}, ..., a_1]$. Then we have the following relations

$$M_{n+1} = P + \frac{1}{a_{n+1}} + \frac{1}{a_n} + \frac{1}{Q}, \qquad M_n = \frac{1}{P} + a_{n+1} + \frac{1}{a_n + Q^{-1}},$$

$$M_{n-1} = a_n + \frac{1}{Q} + \frac{1}{a_{n+1} + P^{-1}}, \qquad M_{n-2} = Q + \frac{1}{a_n} + \frac{1}{a_{n+1}} + \frac{1}{P}.$$

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It is easy to show that the following table is correct, where the arrow $\nearrow$ under P means ascending function in P, $\searrow$ means descending function.

M	P	Q	a_n	a_{n+1}
M_{n+1}	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$
M_n	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$
M_{n-1}	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$
M_{n-2}	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Table 1

In the following, notice that Q is always a rational number, if we replace Q with an irrational number, we can never obtain equality, hence we drop equality sign from $\geqslant$ to prevent confusion.

Lemma 1. *If* $P > (\sqrt{5}+1)/2$, $Q > (\sqrt{5}+1)/2$, $a_n = 1$, $a_{n+1} = 1$, *then at least two of* M_i $(i = n-2, n-1, n, n+1)$ *exceed* $\sqrt{5}$.

Proof. (1) If $Q < P$, then $M_{n+1} > \sqrt{5}$, by Borel theorem at least one of M_i $(i = n-2, n-1, n)$ exceeds $\sqrt{5}$, hence two M's exceed $\sqrt{5}$. (2) If $Q > P$, then $M_{n-2} > \sqrt{5}$, by Borel theorem, one of M_i $(i = n-1, n, n+1)$ exceeds $\sqrt{5}$, hence two M's exceed $\sqrt{5}$.

Lemma 2. *If* $P > (\sqrt{5}+1)/2$, $Q < (\sqrt{5}+1)/2$, $a_n = 1$, $a_{n+1} \geqslant 2$, *then* $M_n > 2.5$, $M_{n-1} > (\sqrt{5}+1)/2 + 1/3$, $M_n + M_{n-1} > 4.5$.

Proof. $M_n > 2.5$, $M_{n-1} > (\sqrt{5}+1)/2 + 1/3$ are trivial. If $a_{n+1} = 2$, from $1/P + 1/(a + P^{-1})$ is descending in P and $a_{n+1} + 1/Q + (a_{n+1} + 1/Q)^{-1} > 2$, we have $M_n + M_{n-1} > 4.5$, for $a_{n+1} \geqslant 3$, $M_n + M_{n-1} > 4.5$ is trivial.

Lemma 3. *If* $P < (\sqrt{5}+1)/2$, $Q < (\sqrt{5}+1)/2$, *then max* $(M_n, M_{n-1}) > \sqrt{5}$, $\min(M_n, M_{n-1}) > (\sqrt{5}+2)/2$, $M_n + M_{n-1} > 2\sqrt{5}$.

Proof. If $P < Q$, then $M_n > \sqrt{5}$ while $M_{n-1} > 1 + 1/Q + 1/2 > (\sqrt{5}+2)/2$. Similarly, if $P > Q$ we have $M_{n-1} > \sqrt{5}$, $M_n > (\sqrt{5}+2)/2$. Since $M_n + M_{n-1} = 2 + 1/P + 1/(1 + P^{-1}) + 1/Q + 1/(1 + Q^{-1})$ is decreasing in both P and Q, hence $M_n + M_{n-1} > 2\sqrt{5}$.

Theorem 1. *In four consecutive M_i ($i = n-1, n-2, n, n+1$), one of the following statement:*

(1) *at least two M's exceed $\sqrt{5}$,*

(2) *at least one M exceeds 3, and another M exceeds $(\sqrt{5}+1)/2$,*

(3) *the sum of two M's exceeds $2\sqrt{5}$, one of these two M's exceeds $\sqrt{5}$, the other exceeds $(\sqrt{5}+1)/2+1/3$.*

Proof. We examine the following 16 cases, in each case (except (1), (4), (6), (13)), it is easy to check the conclusion by using Table 1. For simplicity we write $\omega = (\sqrt{5}+1)/2$.

(1) $P > \omega$, $Q > \omega$, $a_n = 1$, $a_{n+1} = 1$ (cfr. Lemma 1).

(2) $P > \omega$, $Q > \omega$, $a_n = 1$, $a_{n+1} \geqslant 2$. We have $M_n > (\sqrt{5}+3)/2$, $M_{n-2} > (\sqrt{5}+2)/2$.

(3) $P > \omega$, $Q > \omega$, $a_n \geqslant 2$, $a_{n+1} = 1$. We have $M_{n-1} > (\sqrt{5}+3)/2$, $M_{n+1} > (\sqrt{5}+2)/2$.

(4) $P > \omega$, $Q > \omega$, $a_n \geqslant 2$, $a_{n+1} \geqslant 2$. If a_n or a_{n+1} exceeds 3, then M_{n-1} or M_n exceeds 3, while M_{n+1} exceeds $(\sqrt{5}+1)/2$.

(5) $P > \omega$, $Q < \omega$, $a_n = 1$, $a_{n+1} = 1$. We have M_{n+1}, $M_{n-1} > \sqrt{5}$.

(6) $P > \omega$, $Q < \omega$, $a_n = 1$, $a_{n+1} \geqslant 2$ (cfr. Lemma 2).

(7) $P > \omega$, $Q < \omega$, $a_n \geqslant 2$, $a_{n+1} = 1$. We have M_{n+1}, $M_{n-1} > \sqrt{5}$.

(8) $P > \omega$, $Q < \omega$, $a_n \geqslant 2$, $a_{n+1} \geqslant 2$. We have $M_{n-1} > (\sqrt{5}+3)/2$, $M_n > 2$.

(9) $P < \omega$, $Q > \omega$, $a_n = 1$, $a_{n+1} = 1$. We have M_{n-2}, $M_n > \sqrt{5}$.

(10) $P < \omega$, $Q > \omega$, $a_n = 1$, $a_{n+1} \geqslant 2$. We have M_{n-2}, $M_n > \sqrt{5}$.

(11) $P < \omega$, $Q > \omega$, $a_n \geqslant 2$, $a_{n+1} = 1$. We have M_{n-1}, $M_{n+1} > \sqrt{5}$.

(12) $P < \omega$, $Q > \omega$, $a_n \geqslant 2$, $a_{n+1} \geqslant 2$. We have $M_n > (\sqrt{5}+3)/2$, $M_{n-1} > 2$.

(13) $P < \omega$, $Q < \omega$, $a_n = 1$, $a_{n+1} = 1$ (cfr. Lemma 3).

(14) $P < \omega$, $Q < \omega$, $a_n = 1$, $a_{n+1} \geqslant 2$. Similar to case (6). We have $M_n > 2.5$, $M_{n-1} > (\sqrt{5}+1)/2+1/3$, $M_n + M_{n-1} > 4.5$.

(15) $P < \omega$, $Q < \omega$, $a_n \geqslant 2$, $a_{n+1} = 1$. Similar to case (6). We have $M_{n-1} > 2.5$, $M_n > (\sqrt{5}+1)/2+1/3$, $M_n + M_{n-1} > 4.5$.

(16) $P < \omega$, $Q < \omega$, $a_n \geqslant 2$, $a_{n+1} \geqslant 2$. We have M_n, $M_{n-1} > \sqrt{5}$.

Corollary. *In four consecutive M_i ($i = n-2, n-1, n, n+1$) either two M's exceed $\sqrt{5}$ or the sum of two M's exceeds $2\sqrt{5}$.*

Now we use the method in Theorem 1 to improve slightly Brauer and Macon's theorem.

Theorem 2. *In five consecutive M_i $(i = n-2, n-1, n, n+1, n+2)$, either two M's exceed $\sqrt{5}$ or $M_n > 3$ and $M_{n-1}+M_n+M_{n+1} > (1975+147\sqrt{5})/330 = 6.98 > 3\sqrt{5}$.*

Proof. If there are not two M's exceeding $\sqrt{5}$, by Brauer and Macon's theorem, there is an M exceeding 3, this M must be M_n, otherwise we can find another M exceeding $\sqrt{5}$ from the other three or four consecutive M's.

Let $P = [a_{n+3}; a_{n-4}, \ldots]$, $Q = [a_{n-1}; a_{n-2}, \ldots]$ then we can write

$$M_{n+2} = P + 1/a_{n+2} + 1/a_{n+1} + 1/a_n + 1/Q,$$

$$M_{n+1} = a_{n+2} + \frac{1}{a_{n+1}} + \frac{1}{a_n} + \frac{1}{Q} + \frac{1}{P}, \quad M_n = a_{n+1} + \frac{1}{a_n + Q^{-1}} + \frac{1}{a_{n+2} + P^{-1}},$$

$$M_{n-1} = a_n + \frac{1}{Q} + \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \frac{1}{P}, \quad M_{n-2} = Q + \frac{1}{a_n} + \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \frac{1}{P}.$$

From $M_n > 3$, we know that $a_{n+1} \geqslant 2$, denote $s = M_{n-1} + M_n + M_{n+1}$, then

(1) If $a_{n+1} \geqslant 3$, we have $s > 7$.

(2) If $a_{n+1} = 2$, $a_n \geqslant 2$, $a_{n+2} \geqslant 2$, we have $s > 7$.

(3) If $a_{n+1} = 2$, $a_n = 1$, $a_{n+2} \geqslant 2$, we have $s > M_n + 1 + 2 + 3^{-1} + P^{-1} + Q^{-1} + 3^{-1}$, but from M_{n+2}, $M_{n-2} < \sqrt{5}$, we know $P, Q < \sqrt{5}$, hence $s > 7$.

(4) If $a_{n+1} = 2$, $a_n \geqslant 2$, $a_{n+2} = 1$, similar to (3), we have $s > 7$.

(5) If $a_{n+1} = 2$, $a_n = 1$, $a_{n+2} = 1$, then from M_{n+2}, $M_{n-2} < \sqrt{5}$, we know that $P, Q < 5 - 5/7$, hence $s > M_n + 1 + 1 + 3^{-1} + 3^{-1} + 2/(5 - 5/7) > (1975 + 147\sqrt{5})/330$.

References

[1] E. Borel, *Contribution a l'analyse arithmétique du continu*, J. Math. Pures Appl. **9** (1903), 329-375.

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Abstract

Let α be a real number, $[a_0; a_1, a_2, \ldots]$ be its simple continued fraction expansion, p_n/q_n be its nth convergent. Let $|\alpha - p_n/q_n| = 1/(M_n q_n^2)$. In this paper we prove a generalization of Borel theorem. In four consecutive M_i $(i = n-2, n-1, n, n+1)$, either two M's exceed $\sqrt{5}$ or the sum of two M's exceeds $2\sqrt{5}$.

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