Riv. Mat. Univ. Parma (4) 9 (1983), 15-25

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The alternative method for boundary value problems with ordinary differential equations (**)

In a series of papers, Cesari $[\mathbf{1}]_{1,2,3,4}$, Locker $[\mathbf{6}]_{1,2}$ and Hale [4] developed a process or alternative method, based on functional analysis, for the determination of harmonic and subharmonic solutions of nonlinear ordinary differential systems, and for nonlinear boundary value problems for ordinary and partial differential equations (selfadjoint or not selfadjoint, at resonance or not at resonance) with nonlinearities which need not be small.

The method, denoted in [4] as the *alternative method*, has been developed on theoretical lines by many authors. Let us mention here Knobloch $[\mathbf{5}]_{1,2}$ and Locker $[\mathbf{6}]_{1,2}$ for ordinary differential equations, Osborn and Salther for ordinary and elliptic boundary value problems under monotonicity hypotheses, Landesman, Lazer, Williams and Cesari for elliptic boundary value problems, and Cesary and Kannan for hyperbolic problems. We just mention here that Harris, Sibuya, and Weinberg used the method for straitforward proofs of the theorems of Cauchy, Frobenius, Perron, Lettenmeyer in linear differential equations theory in the complex field, and Cesari for a direct proof of Kowalewsky's theorem for partial differential equations. We refer to $[\mathbf{1}]_5$ for the large bibliography. We recall here from this bibliography, that, in the way of exemples, Cesari $[\mathbf{1}]_1$ proved the existence of a solution to the equation $x'' + x^3 = \sin t$, $0 \leqslant t \leqslant 2\pi$, with boundary conditions $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$, (a selfadjoint problem at resonance), and that Locker $[\mathbf{6}]_{1,2}$, Cesari $[\mathbf{1}]_{2,3}$, Cesari and Bowman, and Bononcini considered other particular problems for ordinary and partial differential equations.

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In this paper we first present in **1** a few points of the theory of selfadjoint linear differential operators from [**2**]. In **2** we deal with the general framework of the alternative method for selfadjoint boundary value problems of nonlinear ordinary differential equations of order n based on **1** and on the work of Cesari [**1**]$_1$ and Locker [**6**]$_{1,2}$. Finally ,we indicate a numerical problem of the form $x''+x^3=\varphi(t)$, $0\leqslant t\leqslant 1$, with $x(0)=0$, $x(1)=0$ (a selfadjoint problem, not at resonance). Elsewhere we shall apply the theory to show that this problem has indeed a solution. (Here $Lx=x''$, $\ker L=0$, and yet we shall decompose the underlying space $S=L_2[0,1]$ into orthogonal spaces $S=S_0+S_1$, S_0 of dimension 2. We take for φ a function which belongs to S_1, the argument we shall use is the same as in Cesari's paper [**1**]$_1$, and we shall show that the case $\varphi\in S_1$ brings some simplifications in the argument).

1 - Preliminaries

Let L be the nth order differential operator given by

$$\mathrm{L}x=p_0x^{(n)}+p_1x^{(n-1)}+p_2x^{(n-2)}+\ldots+p_nx\,,$$

where the p_j's are complex-valued functions of class $C^{n-j}_{ap}[a,b]$ and $p_0(t)\neq 0$ on $[a,b]$. Let $U_j(x)$ be the linear form

$$U_j(x)=\sum_{k=1}^{n}M_{jk}x^{(k-1)}(a)+N_{jk}x^{(k-1)}(b)\,,$$

where the M_{jk} and N_{jk} are constants, and let us denote the relationships $U_jx=0$, $j=1,2,\ldots,n$, by $\mathrm{U}x=0$. The problem π: $\mathrm{L}x=lx$, $\mathrm{U}x=0$ is called an eigenvalue problem. It is said to be self-adjoint if $(\mathrm{L}u,v)=(u,\mathrm{L}v)$ for all $u,v\in C^n[a,b]$ which satisfy the boundary conditions $\mathrm{U}u=\mathrm{U}v=0$.

Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote

$$(f,g)=\int_a^b f\bar{g}\,\mathrm{d}t\,,\qquad \|f\|=(\int_a^b |f|^2\,\mathrm{d}t)^{1/2}\,,$$

for $f,g\in L_2[a,b]$. Let us denote by S the space $L_2[a,b]$. If $(f,g)=0$, then f and g are said to be orthogonal. The problem π always has the trivial solution. If l is such that π has a nontrivial solution, then l is called an eigenvalue of π and the nontrivial solutions of π corresponding to this l are called eigenfunctions. We need the following results (see [**2**], pp. 192-201).

(A) Let the problem π be selfadjoint. Then the eigenvalues are real and constitute at most a denumerable set with no finite cluster point. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Consider the non-homogeneous problem

$$\text{(1)} \qquad \mathrm{L}x = lx + f\,, \qquad \mathrm{U}x = 0\,, \qquad \text{where } f \in C[a, b]\,.$$

(B) If at least for one value of l the problem π has no solution except the trivial solution (which is always true for the selfadjoint case), then there exists a unique function $G = G(t, \tau, l)$ defined for (t, τ) on the square $a < t$, $\tau \leqslant b$ and for all complex l except the eigenvalues of π, with the following properties:

(i) $\partial^k G/\partial t^k$, $k = 0, 1, \ldots, n-2$, exist and are continuous in (t, τ, l) for (t, τ) on the square $a \leqslant t$, $\tau \leqslant b$ and l not an eigenvalue of π. Moreover, $\partial^k G/\partial t^k$ for $k = n-1$ and n are continuous in (t, τ, l) for (t, τ) on each of the triangles $a \leqslant t \leqslant \tau \leqslant b$ and $a \leqslant \tau \leqslant t \leqslant b$ and l not at an eigenvalue of π. For fixed (t, τ) these functions are all meromorphic functions of l.

(ii) $\partial^{n-1} G/\partial t^{n-1}(\tau + 0, \tau, l) - \partial^{n-1} G/\partial t^{n-1}(\tau - 0, \tau, l) = 1/p_0(\tau)$.

(iii) As a function of t, G satisfies $\mathrm{L}x = lx$ if $t \neq \tau$.

(iv) As a function of t, G satisfies the boundary conditions $\mathrm{U}x = 0$ for $a \leqslant \tau \leqslant b$.

The solution of (1) is given by the function u defined by

$$u(t) = \int_a^b G(t, \tau) f(\tau)\, \mathrm{d}\tau\,.$$

The function G is known as Green's function for π. Let us assume now, for a moment, with Coddington and Levinson [**2**], that $l = 0$, and that 0 is not an eigenvalue of the selfadjoint problem π. At the end of this section we shal. consider the case where 0 is an eigenvalue.

Since $l = 0$ is not an eigenvalue of π, $G(t, \tau, 0)$ exists. In the rest of our considerations the Green's function for $l = 0$ will be denoted by $G = G(t, \tau)$ and it would be assumed that π is selfadjoint.

Corresponding to this Green's function G, let the linear integral operator H be defined for all $f \in C[a, b]$ by

$$\mathrm{H}f(t) = \int_a^b G(t, \tau) f(\tau)\, \mathrm{d}\tau\,.$$

If $f, g \in C[a, b]$, then $(Lu, v) = (u, Lv)$ applied to $u = Hf$, $v = Hg$ yields $(f, Hg) = (Hf, g)$. From this it follows that (Hf, f) is real.

The operator H is inverse of the operator L in the sense that $LHf = f$, $HLu = u$ are valid for all $f \in C[a, b]$, and $u \in C^n[a, b]$ for which $Uu = 0$.

(C) The eigenfunctions of H are identical with those of π and the eigenvalues of H are reciprocals of those of π.

(D) H is a completely continuous operator.

(E) Either $\|H\|$ or $-\|H\|$ is an eigenvalue for H, and there are an infinite number of eigenvalues $\mu_1, \mu_2, \ldots,$ and eigenfunctions $x_1, x_2, \ldots.$ Moreover

$$|\mu_1| \geqslant |\mu_2| \geqslant \ldots,\ |\mu_k| \to 0 \quad \text{as } k \to \infty,\ \sum_{k=1}^{\infty} |\mu_k|^2 < \infty .$$

Also, x_k $(k = 1, 2, \ldots)$ can be assumed to be an orthonormal sequence which is uniformly bounded on $[a, b]$.

Finally, for $\lambda_k = 1/\mu_k$, we have $|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots,$ and

$$G(t, \tau) = \sum_{k=1}^{\infty} \lambda_k^{-1} x_k(t) x_k(\tau) .$$

(F) If $f \in L_2[a, b]$, then $f = \sum_{k=1}^{\infty} (f, x_k) x_k$, where the equality is understood in the mean sense i.e. $\lim_{m \to \infty} \|f - \sum_{k=1}^{m} (f, x_k) x_k\| = 0$.

Furthermore, Parseval's equality holds $\|f\|^2 = \sum_{k=1}^{\infty} |(f, x_k)|^2$.

Let us consider now the case we have omitted before, where $l = 0$ and 0 is an eigenvalue of π, say of multiplicity m_0, $1 \leqslant m_0 < \infty$, in the sense that there are m_0 independent eigenfunctions x_k with $Lx_k = 0$, $Ux_k = 0$ $(k = 1, \ldots, m_0)$. Let m be any integer, $m \geqslant m_0$, and let $x_{m_0+1}, \ldots, x_m$ be $m - m_0$ more independent eigenfunctions relative to the eigenvalues, say, $\lambda_{m_0+1}, \ldots, \lambda_m$, $0 < |\lambda_{m_0+1}| \leqslant \ldots \leqslant |\lambda_m|$, that is, $Lx_k = \lambda_k x_k$, $Ux_k = 0$ $(k = m_0 + 1, \ldots, m)$. We may well assume that $x_1, \ldots, x_m$ are orthonormal in $L_2[a, b]$. Let $S_0 = \mathrm{sp}(x_1, \ldots, x_m)$ denote the span of $x_1, \ldots, x_m$, and let $P\colon S \to S$ be the orthogonal projection of S onto S_0. Then P is defined by $Pf = \sum_{k=1}^{m} (f, x_k) x_k$ for all $f \in S$. Let $S_1 = S_0^{\perp}$ in S. Then S has the decomposition $S = S_0 + S_1$, where $S_1 = (I\text{-}P)\ S$ is the Hilbert space of all elements $f \in S$ with $(f, x_k) = 0$ $(k = 1, \ldots, m)$, or equivalently, S_1 is the closure in the norm $\|\cdot\|$ of the elements $y \in C[a, b]$ with

$(y, x_k) = 0$ $(k = 1, \ldots, m)$. Also, the operator L, restricted to S_1, has the eigenvalues $\lambda_{m+1}, \lambda_{m+2}, \ldots$, with $0 < |\lambda_{m+1}| \leqslant |\lambda_{m+2}| \leqslant \ldots$, so that 0 is not an eigenvalue of L restricted to S_1. Thus,

$$G(t, \tau) = \sum_{k=m+1}^{\infty} \lambda_k^{-1} x_k(t) x_k(\tau)$$

is the Green function of L in S_1, and H: $S_1 \to S_1$, the operator inverse to $\mathrm{L}|S_1$, is now defined by

$$\mathrm{H}f(t) = \int_a^b G(t, \tau) f(\tau)\, \mathrm{d}\tau \qquad \text{for } f \in S_1 = (\mathrm{I}\text{-}\mathrm{P})S \, .$$

Now every $f \in S$ has the decomposition $f = f_0 + f_1$, with

$$f_0 = \mathrm{P}f = \sum_1^m (f, x_k) x_k \, , \qquad f_1 = (\mathrm{I}\text{-}\mathrm{P})f = \sum_{m+1}^{\infty} (f, x_k) x_k ;$$

hence $f = \sum_{k=1}^{\infty} (f, x_k) x_k$ and (F) holds for any $f \in S$. Also, $\mathrm{LH}f = f$, $\mathrm{HL}u = u$, for all $f \in C[a, b] \cap S_1$, and $u \in C^n[a, b] \cap S_1$ for which $\mathrm{U}u = 0$. Hence, we also have

$$\mathrm{LH}(\mathrm{I}\text{-}\mathrm{P})f = (\mathrm{I}\text{-}\mathrm{P})f \, , \qquad \mathrm{HL}(\mathrm{I}\text{-}\mathrm{P})u = (\mathrm{I}\text{-}\mathrm{P})u \, ,$$

for all $f \in C[a, b]$, and $u \in C^n[a, b]$ for which $Uu = 0$.

2 - Solution of the boundary value problem

Let

$$L = p_0 D^n + p_1 D^{n-1} + \ldots + p_n \, , \qquad p_k \in C^{n-k}[a, b] \, .$$

Let the problem π in **1** be selfadjoint, and let U denote the relative linear form.

Let us consider the nonlinear boundary value problem of order n,

$$\mathrm{L}x = \mathrm{q}(x, t) \, , \qquad \mathrm{U}x = 0 \, , \tag{2}$$

where q is defined for $|x| \leqslant R$, $t \in [a, b]$. We suppose that there are certain functions $K(t)$, $a \leqslant t \leqslant b$, $K \in L_2[a, b]$ and $\eta(\xi)$, $\xi \geqslant 0$, continuous and monotone with $\eta(0) = 0$, such that

$$|q(x, t)| \leqslant K(t)\,, \tag{3}$$
$$|q(x_1, t) - q(x_2, t)| \leqslant \eta(|x_1 - x_2|)K(t)\,,$$

for all $|x_1|$, $|x_2| \leqslant R$, $a \leqslant t \leqslant b$.

Let us take $K_0 = (\int_a^b K^2(t)\,dt)^{1/2}$ and let $S = L_2[a, b]$.

Let x_k ($k = 1, 2, \ldots$) be the eigenfunctions of the operator π relative to the eigenvalues $\lambda_1, \lambda_2, \ldots$, with $0 \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \ldots$. If 0 is an eigenvalue, say of multiplicity m_0, then $0 = \lambda_1 = \ldots = \lambda_{m_0} < |\lambda_{m_0+1}| \leqslant \ldots$. For any $f \in S$ we have

$$f(t) = \sum_{k=1}^{\infty} b_k x_k\,, \qquad b_k = (f, x_k) \qquad (k = 1, 2, \ldots)\,.$$

By Parseval equality we have now

$$\|f\| = (\sum_{k=1}^{\infty} |b_k|^2)^{1/2}\,.$$

For any integer $n \geqslant m_0$ let us define the operator $\mathrm{P}\colon S \to S$ by

$$\mathrm{P}f = \sum_{k=1}^{m} b_k x_k\,.$$

Obviously, $\mathrm{P}^2 = \mathrm{P}$, $\|\mathrm{P}f\| = (\sum_{k=1}^{m} |b_k|^2)^{1/2} \leqslant \|f\|$, i.e., P is a bounded linear projection of S onto the subspace $S_0 = \mathrm{sp}\,(x_1, x_2, \ldots, x_m)$, the space spanned by $x_1, x_2, \ldots, x_m$, and $\|\mathrm{P}\| = 1$. Also, we take $S_1 = S_0^{\perp} = [f \in S, Pf = 0]$, so that we have the decomposition $S = S_0 + S_1$. For any $f \in S_1$ we define the linear operator $\mathrm{H}\colon S_1 \to S_1$ by taking $\mathrm{H}f(t) = \int_a^b G(t, \tau)f(\tau)\,d\tau$ for $f \in S_1$, where $G(t, \tau)$ is the Green's function corresponding to $l = 0$ and the restriction of L to S_1.

Obviously, H is well defined on S_1. Now, for $f \in S_1$ we have

$$(4)\qquad \begin{aligned} \|Hf\| &= \|\int_a^b G(t,\tau) f(\tau)\, d\tau\| \\ &= \|\int_a^b \sum_{k=m+1}^{\infty} \lambda_k^{-1} x_k(t) x_k(\tau) f(\tau)\, d\tau\| \\ &= \|\int_a^b \Big(\sum_{k=m+1}^{\infty} \lambda_k^{-1} x_k(t) x_k(\tau)\Big)\Big(\sum_{k=m+1}^{\infty} b_k x_k(\tau)\Big)\, d\tau\| \\ &= \|\sum_{k=m+1}^{\infty} \lambda_k^{-1} b_k x_k(t)\| = \Big(\sum_{k=m+1}^{\infty} |\lambda_k|^{-2} |b_k|^2\Big)^{1/2} \\ &\leqslant |\lambda_{m+1}|^{-1} \Big(\sum_{k=m+1}^{\infty} |b_k|^2\Big)^{1/2} \leqslant |\lambda_{m+1}|^{-1} \|f\| , \end{aligned}$$

i.e.,
$$\|Hf\| \leqslant |\lambda_{m+1}|^{-1} \|f\| .$$

Analogously, if M is the uniform bound for the eigenfunctions x_k, we have

$$(5)\qquad \begin{aligned} Hf(t)| &= |\int_a^b G(t,\tau) f(\tau)\, d\tau| = |\sum_{k=m+1}^{\infty} \lambda_k^{-1} b_k x_k(t)| \\ &\leqslant M \sum_{k=m+1}^{\infty} |\lambda_k|^{-1} |b_k| \leqslant M \Big(\sum_{k=m+1}^{\infty} |\lambda_k|^{-2}\Big)^{1/2} \|f\| = \sigma M \sigma(m) \|f\| , \end{aligned}$$

where $\sigma^2(m) = \sum_{k=m+1}^{\infty} |\lambda_k|^{-2}$, and $\sigma(m) \to 0$ as $m \to \infty$. Thus, $|Hf(t)| \leqslant M\sigma(m)\|f\|$. The reader may compare (4) and (5) with the analogous relations in Cesari's paper **[1]**$_1$.

For any $f \in S$, $f - Pf \in S_1$, and hence

$$\|H(f - Pf)\| \leqslant |\lambda_{m+1}|^{-1} \|f\| , \qquad |H(f - Pf)| \leqslant M\sigma(m) \|f\| .$$

For any given $R > 0$ we shall denote by S_R the set $S_R = [f \in S,\ |f| \leqslant R]$. Let us now define the operators q, h, F in S_R as follows

$$\begin{aligned} qx &= q(x(t), t) & x \in S_R , \\ hx &= qx - Pqx & x \in S_R , \\ Fx &= H(qx - Pqx) & x \in S_R . \end{aligned}$$

Clearly, for every $x \in S_R$, we have $\mathrm{h}x \in S_1$, $\mathrm{F}x \in S_1$. Hence,

(6) $$\mathrm{PF}x = 0\,, \quad \mathrm{Hh}x = \mathrm{H(I\text{-}P)\,q}x\,, \quad x \in S_R\,.$$

One proves as in [1]$_1$ that there exists a monotone continuous function $\eta^*(\xi)$, $\xi \geqslant 0$, $\eta^*(0) = 0$, such that

(7) $$\|q(x_1, t) - \mathrm{q}(x_2, t)\| \leqslant \eta^*(\|x_1 - x_2\|)$$

for all $x_1, x_2 \in S_R$. We now have by (7)

$$\|\mathrm{h}x_1 - \mathrm{h}x_2\| = \|\mathrm{q}x_1 - \mathrm{q}x_2 - \mathrm{P)q}x_1 - \mathrm{q}x_2)\| \leqslant \|\mathrm{q}x_1 - \mathrm{q}x_2\| \leqslant \eta^*(\|x_1 - x_2\|)\,.$$

Finally, by (4) and (5) we also have

(8) $$\begin{aligned}\|\mathrm{F}x_1 - \mathrm{F}x_2\| &= \|\mathrm{H}(\mathrm{h}x_1 - \mathrm{h}x_2)\| \\ &\leqslant |\lambda_{m+1}|^{-1}\|\mathrm{h}x_1 - \mathrm{h}x_2\| \leqslant |\lambda_{m+1}|^{-1}\eta^*(\|x_1 - x_2\|)\,,\end{aligned}$$

(9) $$|\mathrm{F}x_1 - \mathrm{F}x_2| \leqslant M\sigma(m)\eta^*(\|x_1 - x_2\|)\,.$$

Note that $\|\mathrm{q}x\| \leqslant K_0$ for some constant K_0, and hence $\|\mathrm{h}x\| \leqslant \|\mathrm{q}x\| \leqslant K_0$, and

(10) $$\|\mathrm{F}x\| \leqslant |\lambda_{m+1}|^{-1}K_0\,.$$

If we denote by $\mu(x)$ the norm $\mu(x) = \text{ess sup } x(t)$ for $x \in S$, then the restriction $|x(t)| \leqslant R$ for all $t \in [a, b]$ can be rewritten as $\mu(x) \leqslant R$, and thus S_R can be thought of as the subset of all such x with $\mu(x) \leqslant R$.

Now for every $x \in S_R$, we have

(11) $$|\mathrm{F}x| = |\mathrm{H}(\mathrm{q}x - \mathrm{Pq}x)| \leqslant M\sigma(m)\|\mathrm{q}x\| \quad \text{and} \quad |\mathrm{F}x| \leqslant M\sigma(m)K_0\,.$$

For given constants $c, r > 0$, let x^* be any element of S_0 with

(12) $$\|x^*\| \leqslant c\,, \qquad |x^*| \leqslant r \quad (r < R)\,.$$

(13) Let $$S_R^* = \{x \,|\, x \in S, \mathrm{P}x = x^*, \|x\| \leqslant d, |x| \leqslant R\}\,,$$

where d is another constant with $d > c$. We may assume m large enough so that

(14) $$|\lambda_{m+1}|^{-1}K_0 \leqslant d - c\,, \quad M\sigma(m)K_0 \leqslant R - r\,,$$

and this is possible since both $|\lambda_{m+1}|^{-1}$ and $\sigma(m)$ approach zero as $m \to \infty$. Now let us consider the map T, or $y = \mathrm{P}x + \mathrm{F}x = \mathrm{T}x$ on S_R^*. Obviously, $\mathrm{P}y = \mathrm{PT}x$, $\mathrm{P}(\mathrm{P}x + Fx) = \mathrm{P}x + \mathrm{PF}x = \mathrm{P}x = x^*$, and by (10), (12) and (14) we have

$$\|y\| = \|\mathrm{P}x + \mathrm{F}x\| \leqslant \|\mathrm{P}x\| + \|\mathrm{F}x\| \leqslant c + d - c = d\,.$$

Analogously, by (11), (12) and (14) we also have

$$|y| = |\mathrm{P}x + \mathrm{F}x| \leqslant |\mathrm{P}x| + |\mathrm{F}x| \leqslant r + R - r = R\,.$$

This shows that $\mathrm{T}\colon S_R^* \to S_R^*$. Obviously, the set S_R^* is closed, bounded and convex. We can also show that $\mathrm{T}(S_R^*)$ is relatively compact. Indeed, let us take any sequence $y_k \in \mathrm{T}(S_R)$. Then there are $x_k \in S_R$ such that $y_k = \mathrm{P}x_k + \mathrm{F}x_k = x^* + \mathrm{H}(\mathrm{I}\text{-}\mathrm{P})qx_k$.
As we noticed in **1**, H is completely continuous, I-P is bounded, and hence H(I-P) is completely continuous. Therefore, $\mathrm{H}(\mathrm{I}\text{-}\mathrm{P})\mathrm{q}x_k$ have a convergent subsequence. This readily implies that the sequence $[y_n]$ has a convergent subsequence. Hence, $\mathrm{T}(S_R)$ is relatively compact in the topology of $L_2[a, b]$.

Now by Schauder's principle there is at least one fixed point of T, that is, there exists $y \in S_R$ such that $y = x^* + \mathrm{F}y$ with $\mathrm{P}y = \mathrm{P}x^*$, and $y(t) = x^* + H(\mathrm{q}y - \mathrm{Pq}y)$.

Since $\mathrm{P}x^*$ is a finite linear combination of x_k's and $\mathrm{U}G(t, \tau) = 0$, we have

$$\mathrm{U}y = \mathrm{U}x^* + \mathrm{UH}(\mathrm{q}y - \mathrm{Pq}y)\,, \tag{15}$$

that is, y satisfies the given boundary conditions. We have now

$$\begin{aligned} y(t) &= \mathrm{P}y + \mathrm{H}(\mathrm{q}y - \mathrm{Pq}y) = \mathrm{P}x^* + \mathrm{H}(\mathrm{q}y - \mathrm{Pq}y)\,,\\ \varphi &= \mathrm{H}(\mathrm{q}y - \mathrm{Pq}y) = \int_a^b G(t, \tau)\big(\mathrm{q}y(\tau) - \mathrm{Pq}y(\tau)\big)\,d\tau\,. \end{aligned} \tag{16}$$

Since $\mathrm{q}y \in L_2[a, b]$, then $\varphi = \mathrm{H}(\mathrm{q}y - \mathrm{Pq}y) \in C^{n-1}[a, b]$, and $\varphi^{(n-1)}$ is absolutely continuous. By applying L on both sides of (16) we have almost everywhere in $[a, b]$

$$\mathrm{L}y = \mathrm{LP}y + \mathrm{LH}(\mathrm{q}y - \mathrm{Pq}y) = \mathrm{PL}y + \mathrm{q}y - \mathrm{Pq}y\,.$$

Thus, $\mathrm{L}y = \mathrm{q}y + \mathrm{P}(\mathrm{L}y - \mathrm{q}y)$. In other words, $y(t)$ satisfies the nth order equation

$$\text{(17)} \qquad \mathrm{L}y = \mathrm{q}(y(t), t) + D\,,$$

where $D = \mathrm{P}(\mathrm{L}y - \mathrm{q}(y(t), t))$, and D is a Fourier sum of order $m+1$.

Thus, in view of (15), $y(t)$ is a solution of the original boundary value problem (2), say $y(t) = x(t)$, provided

$$\text{(18)} \qquad \mathrm{P}[\mathrm{L}y - \mathrm{q}(y(t), t)] = 0\,,$$

or equivalently, provided

$$\text{(19)} \qquad V_j = \int_a^b [\mathrm{L}y - \mathrm{q}(y(t), t)]\,\bar{x}_j(t)\,\mathrm{d}t = 0 \qquad (j = 1, 2, \ldots, m)\,.$$

Assuming $y(t) = \sum_{k=1}^{\infty} b_k x_k$, we have $\mathrm{L}y(t) = \sum_{k=1}^{\infty} \lambda_k b_k x_k$. Moreover, if $\mathrm{q}(y(t), t) = -\sum_{k=1}^{\infty} c_k x_k$, then $D = \sum_{k=1}^{m} (\lambda_k b_k + c_k) x_k$, and equations (19) take the form

$$\text{(20)} \qquad V_j = \lambda_j b_j + c_j = 0 \qquad (j = 1, 2, \ldots, m)\,.$$

If these equations are satisfied, then $y(t) = x(t)$ is a solution of (2). Equations (20), are said to be *the bifurcation or determining equations.* Since q is a known function of y, the coefficients c_k in the Fourier expansion of q are functions of the Fourier coefficients of y. We obtain therefore a system of m equations in the variables $b_1, b_2, \ldots, b_m$. In other words we have proved the following statement.

Theorem. *Suppose conditions* (3) *and* (14) *hold, then for every* $x^* \in S$ *satisfying* (12) *there exists at least one fixed point* y *of the map* $\mathrm{T}\colon S_R^* \to S_R^*$. *This* $y(t)$ *satisfies* (16) *with* D *given by* (17). *If it is possible to choose* x^* *in such away that the determining equations* (20) *are satisfied, then* $y(t) = x(t)$ *is a solution of the boundary value problem* (2).

Remark. To obtain an approximation, say $x^m(t)$ to the solution $x(t)$ of (2), we may replace $y(t)$ by $x^m(t) = x^*(t)$ in (19). Then system (19) becomes

$$\text{(21)} \qquad v_j = \int_a^b (Lx^{(m)} - q(x^{(m)}(t), t)\,\bar{x}_j(t)\,\mathrm{d}t = 0 \qquad (j = 0, 1, \ldots, m)\,,$$

where $x^* = \sum_{k=1}^{m} b_k x_k$ with undetermined coefficients $b_1, b_2, \dots, b_m$. Equations (21) are the usual equations for the mth-approximation $x^{(m)}(t)$ of the Galerkin' method. If one puts

$$q(x^*(t), t) = -\sum_{k=1}^{\infty} d_k x_k ,$$

then equations (21) become

$$v_j = \lambda_j b_j + d_j = 0 \qquad (j = 1, 2, \dots, m) . \tag{22}$$

We shall denote by $x^{(m)}$ the function $x^*(t)$ when its coefficients satisfy (22).

Elsewhere we shall apply the general process discussed above to a numerical problem. Namely we shall show that the problem

$$-x'' = x^3 + t^2 - (\sqrt{2}/\pi)(1 - 4/\pi^2) 2 \sin \pi t + (1/\pi) \sin 2\text{p}t , \quad x(0) = 0 , \quad x(1) = 0 ,$$

has indeed a solution. In this problem we take $\text{L}x = -x''$, $U_1(x) = x(0)$, $U_2(x) = x(1)$, $J = [0, 1]$.

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