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On almost-periodic perturbation of exponentially dichotomic abstract differential equations (**)

1 – Consider in a Banach space E the differential equation

(1) $$x'(t) = Ax(t) + f(t) \qquad -\infty < t < \infty ,$$

where the closed linear operator A is the infinitesimal generator of a strongly continuous one-parameter group $T(t)$, $t \in R$. We make the following assumptions:

(i) $f(t)$ is a strongly almost-periodic function: $R \to E$.

(ii) Equation (1) is exponentially-dichotomic (or shortly e-dichotomic)

i.e. there exists positive constants N and a such that

$$\|T(t-s)P_+\|_{L(E)} \leqslant N \exp[-a(s-t)] \qquad s \geqslant t ,$$

$$\|T(t-s)P_-\|_{L(E)} \leqslant N \exp[-a(t-s)] \qquad t \geqslant s ,$$

P_+ and P_- are spectral projections on E (see [**2**] for basic definitions and properties).

We state and prove

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Theorem 1. *Let $x(t)$ be a solution of equation* (1) *and suppose assumptions* (i) *and* (ii) *hold. Then $x(t)$ is strongly almost-periodic.*

Lemma 1. *$x(t)$ admits the integral representation*

$$x(t) = \int_{-\infty}^{t} T(t-\sigma)P_- f(\sigma)\, d\sigma - \int_{t}^{\infty} T(t-\sigma)P_- f(\sigma)\, d\sigma .$$

Proof. If $x(\sigma)$ is a solution of equation (1), then we have

$$x'(\sigma) = Ax(\sigma) + f(\sigma) . \tag{1}$$

Fix $t \in R$ and apply $T(t-\sigma)$ to both members of (1)

$$\begin{aligned} T(t-\sigma)x'(\sigma) &= T(t-\sigma)Ax(\sigma) + T(t-\sigma)f(\sigma) \\ &= AT(t-\sigma)x(\sigma) + T(t-\sigma)f(\sigma) , \end{aligned} \tag{2}$$

because A and $T(t)$ commute on $D(A)$. Using the properties of P_+ and P_-: $I_E = P_+ + P_-$, I_E the identity operator on E; $\theta = P_+P_- = P_-P_+$, θ the null operator on E and integrating (2) from $-\infty$ to t, we get

$$\int_{-\infty}^{t} T(t-\sigma)P_- x'(\sigma)\, d\sigma = \int_{-\infty}^{t} AT(t-\sigma)P_- x(\sigma)\, d\sigma + \int_{-\infty}^{t} T(t-\sigma)P_- f(\sigma)\, d\sigma . \tag{3}$$

Consider the following equality

$$\frac{\partial}{\partial\sigma} T(t-\sigma)x(\sigma) = -AT(t-\sigma)x(\sigma) + T(t-\sigma)x'(\sigma) . \tag{4}$$

Integrate from R to t, with $R < t$,

$$P_- x(t) - T(t-R)P_- x(R) = -\int_{R}^{t} AT(t-\sigma)P_- x(\sigma)\, d\sigma + \int_{R}^{t} T(t-\sigma)P_- x'(\sigma)\, d\sigma .$$

But we have $\|T(t-R)P_- x(R)\|_E \leqslant N \exp[-a(t-R)]$, which shows

$$\lim_{R\to-\infty} \|T(t-R)P_- x(R)\|_E = 0$$

and therefore

$$P_- x(t) = -\int_{-\infty}^{t} AT(t-\sigma)P_- x(\sigma)\, d\sigma + \int_{-\infty}^{t} T(t-\sigma)P_- x'(\sigma)\, d\sigma . \tag{5}$$

Combining (3) and (5), we get

(6) $$P_{-}x(t) = \int_{-\infty}^{t} T(t-\sigma) P_{-} f(\sigma)\, d\sigma .$$

Now integrate (2) from t to $+\infty$

(7) $$\int_{t}^{\infty} T(t-\sigma) P_{+} x'(\sigma)\, d\sigma = \int_{t}^{\infty} A T(t-\sigma) P_{+} x(\sigma)\, d\sigma + \int_{t}^{\infty} T(t-\sigma) P_{+} f(\sigma)\, d\sigma .$$

If we integrate (4) from t to R with $R > t$, we get

$$T(t-R) P_{+} x(R) - P_{+} x(t) = -\int_{t}^{R} A T(t-\sigma) P_{+} x(\sigma)\, d\sigma + \int_{t}^{R} T(t-\sigma) P_{+} x'(\sigma)\, d\sigma .$$

But we have $\| T(t-R) P_{+} x(R) \|_{E} \leqslant N \exp[-a(R-t)]$, which gives

$$\lim_{R \to +\infty} \| T(t-R) P_{+} x(R) \|_{E} = 0$$

and therefore

(8) $$-P_{+} x(t) = -\int_{t}^{\infty} A T(t-\sigma) P_{+} x(\sigma)\, d\sigma + \int_{t}^{\infty} T(t-\sigma) P_{+} x'(\sigma)\, d\sigma .$$

Now (7) and (8) give

(9) $$-P_{+} x(t) = \int_{t}^{\infty} T(t-\sigma) P_{+} x'(\sigma)\, d\sigma .$$

Finally, combining (5) and (9)

$$x(t) = P_{-} x(t) + P_{+} x(t) = \int_{-\infty}^{t} T(t-\sigma) P_{-} f(\sigma)\, d\sigma - \int_{t}^{\infty} T(t-\sigma) P_{+} f(\sigma)\, d\sigma .$$

Both integrals are convergent; in fact we have the bound

$$\begin{aligned} \| x(t) \|_{E} &\leqslant \int_{-\infty}^{t} \| T(t-\sigma) P_{-} f(\sigma) \|_{E}\, d\sigma + \int_{t}^{\infty} \| T(t-\sigma) P_{+} f(\sigma) \|_{E}\, d\sigma \\ &\leqslant N \Big\{ \int_{-\infty}^{t} e^{-a(t-\sigma)} \| f(\sigma) \|_{E}\, d\sigma + \int_{t}^{\infty} e^{-a(\sigma-t)} \| f(\sigma) \|_{E}\, d\sigma \Big\} \\ &\leqslant N \sup_{\sigma \in R} \| f(\sigma) \|_{E} \cdot \frac{2}{a} . \end{aligned}$$

$f(\sigma)$ is bounded as a strongly almost-periodic function. The lemma is proved.

Proof of Theorem 1. Let $\varepsilon > 0$ be given; by almost-periodicity of $f(t)$ we can say for every $\tau \in [a, a + l(\varepsilon)]$ which is an ε-translation of $f(\sigma)$ we have $\sup_{\sigma \in R} \|f(\sigma + \tau) - f(\sigma)\|_E < \varepsilon$. Now

$$x(t+\tau) - x(t) = \int_{-\infty}^{t+\tau} T(t+\tau-\sigma) P_- f(\sigma)\, d\sigma - \int_{t+\tau}^{\infty} T(t+\tau-\sigma) P_+ f(\sigma)\, d\sigma$$
$$- \int_{-\infty}^{t} T(t-\sigma) P_- f(\sigma)\, d\sigma + \int_{t}^{\infty} T(t-\sigma) P_+ f(\sigma)\, d\sigma\,,$$

if we put $s + \tau = \sigma$ in the two first integrals, we obtain

$$x(t+\tau) - x(t) = \int_{-\infty}^{t} T(t-\sigma) P_- f(\sigma+\tau)\, d\sigma - \int_{t}^{\infty} T(t-\sigma) P_+ f(\sigma+\tau)\, d\sigma$$
$$- \int_{-\infty}^{t} T(t-\sigma) P_- f(\sigma)\, d\sigma + \int_{t}^{\infty} T(t-\sigma) P_+ f(\sigma)\, d\sigma$$
$$= \int_{-\infty}^{t} T(t-\sigma) P_- \{f(\sigma+\tau) - f(\sigma)\}\, d\sigma - \int_{t}^{\infty} T(t-\sigma) P_+ \{f(\sigma+\tau) - f(\sigma)\}\, d\sigma\,.$$

A simple computation gives

$$\|x(t+\tau) - x(t)\|_E \leqslant \frac{2N}{a} \sup_{\sigma \in R} \|f(\sigma+\tau) - f(\sigma)\|_E < \frac{2N}{a} \cdot \varepsilon\,,$$

therefore $\sup_{t \in R} \|x(t+\tau) - x(t)\|_E < 2N/a \cdot \varepsilon$, which proves almost-periodicity of $x(t)$.

2 – Let now $A = A(t)$ varies with time; we are going to prove almost-periodicity of solutions of

$$x'(t) = A(t)x(t) + f(t) \qquad -\infty < t < \infty\,. \tag{2}$$

Consider the following assumptions:

(i) $f(t)$ is a strongly almost-periodic function $R \to E$.

(ii) $A(t) \in L(E)$, $\forall t \in R$.

(iii) Equation (2) is e-dichotomic, i.e. there exists positive constants N_1, N_2, a_1, a_2, such that

$$\|S(t) P_1 S(s)^{-1}\|_{L(E)} \leqslant N_1 \exp[-a_1(t-s)] \qquad t > s\,,$$
$$\|S(t) P_2 S(s)^{-1}\|_{L(E)} \leqslant N_2 \exp[-a_2(s-t)] \qquad s > t\,,$$

with $S(t)$, $t \in R$, the Cauchy operators corresponding to equation (2); P_1 and P_2 a pair of mutually complementary projections. We have $I_E = P_1 + P_2$, $P_1 P_2 = P_2 P_1 = \theta$ and $P_i^2 = P_i$, $i = 1, 2$.

(iv) The Green function considered is

$$G(t, s) = \begin{cases} S(t)P_1 S(s)^{-1} & t > s \\ S(t)P_2 S(s)^{-1} & s > t \end{cases}$$

with properties

$$\frac{\partial G(t, s)}{\partial t} = A(t)G(t, s) \quad \text{and} \quad G(t, t+0) - G(t, t-0) = I_E .$$

Theorem 2. *Let $x(t)$ be a solution of equation* (2) *such that assumptions* (i)-(iv) *hold, then $x(t)$ is almost-periodic.*

Lemma 2. *$x(t)$ admits the integral representation*

$$x(t) = \int_{-\infty}^{t} S(t)P_1 S(\sigma)^{-1} f(\sigma)\, d\sigma - \int_{t}^{\infty} S(t)P_2 S(\sigma)^{-1} f(\sigma)\, d\sigma .$$

Proof. Let $Z(t) \equiv \int_{-\infty}^{\infty} G(t, \sigma)\, d\sigma$, $t \in R$. Then we have

$$\|Z(t)\| \leqslant \int_{-\infty}^{t} \|S(t)P_1 S(\sigma)^{-1} f(\sigma)\| d\sigma + \int_{t}^{\infty} \|S(t)P_2 S(\sigma)^{-1} f(\sigma)\| d\sigma$$

$$\leqslant \int_{-\infty}^{t} N_1 e^{-a_1(t-\sigma)} \|f(\sigma)\| d\sigma + \int_{t}^{\infty} N_2 e^{-a_2(\sigma-t)} \|f(\sigma)\| d\sigma$$

$$\leqslant \sup_{\sigma \in R} \|f(\sigma)\| \{\frac{N_1}{A_2} + \frac{N_2}{A_2}\} < \infty \qquad \forall t \in R .$$

Therefore the integral $Z(t)$ converges uniformly over the real line. We also have

$$V(t) \equiv \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(t, \sigma) f(\sigma)\, d\sigma = \int_{-\infty}^{t} A(t)S(t)P_1 S(\sigma)^{-1} f(\sigma)\, d\sigma - \int_{t}^{\infty} A(t)S(t)P_2 S(\sigma)^{-1} f(\sigma)\, d\sigma ,$$

$$\|V(t)\| = \|\int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(t, \sigma) f(\sigma)\, d\sigma\| \leqslant \int_{-\infty}^{t} N_1 e^{-a_1(t-\sigma)} \|f(\sigma)\| \|A(t)\| d\sigma$$

$$+ \int_{t}^{\infty} N_2 e^{-a_2(\sigma-t)} \|f(\sigma)\| \|A(t)\| d\sigma$$

$$\leqslant \sup_{\sigma \in R} \|f(\sigma)\| \cdot \|A(t)\| \{\frac{N_1}{a_1} + \frac{N_2}{a_2}\},$$

therefore $V(t)$ exists for each $t \in R$. We can conclude

$$Z'(t) = \frac{\partial}{\partial t} \int_{-\infty}^{t} S(t) P_1 S(\sigma)^{-1} f(\sigma)\, d\sigma - \frac{\partial}{\partial t} \int_{t}^{\infty} S(t) P_2 S(\sigma)^{-1} f(\sigma)\, d\sigma$$

$$= G(t, t-0) f(t) + \int_{-\infty}^{t} \frac{\partial}{\partial t} S(t) P_1 S(\sigma)^{-1} f(\sigma)\, d\sigma$$

$$- G(t, t+0) f(t) - \int_{t}^{\infty} \frac{\partial}{\partial t} S(t) P_2 S(\sigma)^{-1} f(\sigma)\, d\sigma$$

$$= f(t) + \int_{-\infty}^{t} A(t) G(t, \sigma) f(\sigma)\, d\sigma - \int_{t}^{\infty} A(t) G(t, \sigma) f(\sigma)\, d\sigma$$

$$= f(t) + A(t) \{ \int_{-\infty}^{t} G(t, \sigma) f(\sigma)\, d\sigma - \int_{t}^{\infty} G(t, \sigma) f(\sigma)\, d\sigma \}$$

$$= f(t) + A(t) Z(t) ,$$

which proves $Z(t)$ is a solution of equation (2).

Proof of Theorem 2. Let $x(t)$ be a solution of equation (2); as in the proof of Theorem 1, consider τ an ε-translation of $f(t)$. Then we can get very easily the equality

$$x(t+\tau) - x(t) = \int_{-\infty}^{t} S(t) P_1 S(\sigma)^{-1} \{ f(\sigma+\tau) - f(\sigma) \}\, d\sigma - \int_{t}^{\infty} S(t) P_2 S(\sigma)^{-1} \{ f(\sigma+\tau) - f(\sigma) \}\, d\sigma ,$$

therefore

$$\| x(t+\tau) - x(t) \| \leqslant \sup_{\sigma \in R} \| f(\sigma+\tau) - f(\sigma) \| N_1 \{ \int_{-\infty}^{t} e^{-a_1(t-\sigma)}\, d\sigma + N_2 \int_{t}^{\infty} e^{-a_2(\sigma-t)}\, d\sigma \}$$

$$= \sup_{\sigma \in R} \| f(\sigma+\tau) - f(\sigma) \| \{ \frac{N_1}{a_1} + \frac{N_2}{a_2} \} < \varepsilon \{ \frac{N_1}{a_1} + \frac{N_2}{a_2} \} ,$$

$x(t)$ is almost-periodic.

References

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Abstract

We are concerned in this paper with exponentially dichotomic linear differential equations $x'(t) = Ax(t) + f(t)$, $-\infty < t < \infty$, *in a Banach space* E, *with* $f(t)$ *almost-periodic; we prove almost-periodicity of solutions when* A *is the infinitesimal generator of a strongly continuous group* $T(t)$, *and also when* $A = A(t)$ *varies with time and* $A(t) \in L(E)$, $\forall t \in R$.

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