Riv. Mat. Univ. Parma (4) 8 (1982), 161-173

JEAN MARIE MORVAN (*)

Cylindricity (**)

Introduction

In this work, we generalize the usual notion of « cylinder » in Euclidean space. Roughly speaking, we call a strongly cylindrical submanifold, a submanifold of a Riemannian manifold, such that the second foundamental form is cylindrical in one normal direction, (in sense of B. Y. Chen [3]), null in the others, and such that we can define a « Frenet frame », using the derivations of the cylindrical direction.

In the first paragraph, we study the Gauss-Codazzi and the Codazzi-Ricci equations of a cylinder. In the second paragraph, we study the strongly cylindrical submanifolds in space forms. In the third paragraph, we study the strongly cylindrical submanifolds in Kaehler manifolds. In the fourth paragraph, we use this notion of cylindricity to study immersions which are products of immersions, in the case where the first principal normal space has dimension 2.

We shall use the following notations. Let $i: M^n \to \tilde{M}^{n+p}$ be an isometric immersion of a n-dimensional manifold M^n in a $n+p$ dimensional manifold $\tilde{M}^{n+p}$. We denote by TM^n and $T\tilde{M}^{n+p}$ the tangent space of M^n and $\tilde{M}^{n+p}$. ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections on M^n and $\tilde{M}^{n+p}$. $T^\perp M^n$ denotes the normal bundle, and $\nabla^\perp$ the Riemannian connection induced by $\tilde{\nabla}$ on $T^\perp M^n$. σ: $TM^n \times TM^n \to T^\perp M$ is the second foundamental form defined by $\tilde{\nabla}_X Y$

(*) Indirizzo: Faculté des Sciences d'Avignon, Dept. de Math., 33 Rue Louis Pasteur, 84000 Avignon, France.
(**) Ricevuto: 16-VIII-1980.

$= \nabla_X Y + \sigma(X, Y)$, $\forall X, Y \in TM^n$; R and $\tilde{R}$ are the curvature tensors of M^n and $\tilde{M}^{n+p}$. We have the following equations (Gauss-Codazzi-Ricci)

$$\langle \tilde{R}(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle + \langle \sigma(X, W), \sigma(Y, Z)\rangle - \langle \sigma(Y, W), \sigma(X, Z)\rangle,$$

$$\langle \tilde{R}(X, Y), Z, \xi\rangle = \langle (\overline{\nabla}_X \sigma)(Y, Z) - (\overline{\nabla}_Y \sigma)(X, Z), \xi\rangle,$$

$$\forall X, Y, Z, W \in TM^n, \quad \forall \xi \in T^\perp M^n,$$

where $\overline{\nabla}$ is defined by

$$(\overline{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X g(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) .$$

If $\tilde{M}^{n+p}$ is a Kaehler manifold, we denote by J the complex structure. It is well known that

$$\langle \tilde{X}, \tilde{Y}\rangle = \langle J\tilde{X}, J\tilde{Y}\rangle \quad \forall \tilde{X}, \tilde{Y} \in T\tilde{M}^{n+p}, \qquad \tilde{\nabla}_{\tilde{X}} J\tilde{Y} = J\tilde{\nabla}_{\tilde{X}} \tilde{Y} \quad \forall \tilde{X}, \tilde{Y} \in T\tilde{M}^{n+p}.$$

Following [**2**], [**6**], we give the following definitions.

If, at every point $m \in M^n$, $JT_m M^n = T_m M^n$, M^n is called a *holomorphic submanifold.*

If at every point $m \in M^n$, $JT_m M^n \subset T^\perp_m M^n$, M^n is called a *totally real submanifold.*

If at every point $m \in M^n$, $JT^\perp_m M^n \subset T_m M^n$, M^n is called an *antiholomorphic submanifold.*

If there exists a differentiable distribution $\mathcal{D}$: $m \mapsto \mathcal{D}_m \subset T_m M^n$ satisfying the following conditions: (α) $J\mathcal{D} = \mathcal{D}$ at every point, (β) $J\mathcal{D}^\perp \subset T^\perp M^n$ at every point, then M^n is called a C.R. *submanifold.*

We shall now define the notion of *principal normal spaces*, introduced in [**8**], [**4**]$_{1,2}$, and *external curvatures* [**4**]$_{1,2}$.

Def. 1. Let $m \in M$. Let E_{1_m} be the subspace of $T^\perp_m M$ defined by $E_{1_m} = [\operatorname{Im} \sigma]_m$ (i.e. the subspace spanned by $\operatorname{Im} \sigma$). E_1 is called *the first principal normal space.*

By induction, we define the i-th *principal normal space* E_i in the following way. If $\dim E_{i-1}$ is constant on a neighborhood of m, then $E_{i_m} = [\overline{E}_{i_m}]$, where $\overline{E}_{i_m} = \{\eta \in T^\perp_m M / \exists X \in T_m M, \ \exists \xi \in E_{i-1_m}$ such that $\eta = pr_{(\bigoplus_{j<i} E_j)^\perp}(\nabla^\perp_X \xi)_m\}$.

Remarks. $(E_i)_m = \{0\} \Rightarrow (E_{i+1})_m = \{0\}$. By construction, the sum of these spaces is direct.

Def. 2. Let $m \in M$. If $E_{1_m}, E_{2_m}, \ldots, E_{i_m}$ are defined, we call

j-th *external curvature* $(j = 1, \ldots, i)$ *at* $m \in M$, or j-th–*Frenet curvature at* m, the scalars $(k_j)_m^{(M)}$ defined by

$$j = 1:\ (k_1^{(M)})_m = \operatorname*{Sup}_{\substack{X,Y \in TM^n \\ \|X\|=\|Y\|=1}} \|\sigma(X, Y)\| \,,$$

$j \geqslant 2$: by induction, we define first the applications

$$(k_j)_m\colon (E_{j-1})_m \to \boldsymbol{R}^+ \,, \qquad \eta \mapsto \operatorname*{Sup}_{\substack{X \in TM^n \\ \|X\|=1}} \|pr_{\substack{(\oplus E_s)^\perp \\ s<i}} \nabla_X^\perp \eta\| \,;$$

then

$$(k_j^{(M)})_m = \operatorname*{Sup}_{\substack{\eta \in (E_{j-1})_m \\ \|\eta\|=1}} k_j(\eta)_m \,.$$

1 - A definition of a strongly cylindrical submanifold

Def. 1. Let M^n a n-dimensional submanifold of the Euclidean space $E^{n+p}\cdot M^n$ is called a *cylinder* if and only if M^n is the Riemannian product of a curve C and an open set U of the Euclidean space E^{n-1} (i.e. $M^n = C \times U$).

We shall give some obvious properties of a cylinder in the Euclidean space.

(1) $M^n = C \times U$ is flat. Moreover, if M^n is simply connected, complete, then M^n is isometric to $\boldsymbol{E}^n$.

(2) Let T be the tangent vector field of C. It is clear that T is parallel in M^n, and that the second foundamental form of the immersion, associated to M^n, has the following expression

$$\sigma(X, Y) = \alpha\omega(X)\omega(Y)\xi_1 \qquad \forall X,\ Y \in TM^n \,,$$

where ω is the 1-form defined by $\omega(X) = \langle X, T\rangle$, where α is a function C^∞ on M^n, and ξ_1 is a normal vector field on M^n. Moreover, we have a system of equations.

System 1.

If $\alpha \neq 0$, $\quad \nabla_X^\perp \xi_1 = \tau_2(X)\xi_2$,

if $\tau_r \neq 0$, $\quad \nabla_X^\perp \xi_r = \tau_{r+1}(X)\xi_{r+1} - \tau_r(X)\xi_{r-1}$ $(r = 2, \ldots, j-1)$,

if $\tau_j \neq 0$, $\quad \nabla_X^\perp \xi_j = -\tau_j(X)\xi_{j-1}$, where $\tau_2, \ldots, \tau_j$ ($j \leqslant p$) are $j-1$ closed 1-forms proportional to ω, and $\xi_1, \ldots, \xi_j$ are orthonormal unit vector fields in the normal bundle. Clearly, the Frenet frame of the curve is the restriction of $\xi_1, \ldots, \xi_j$ on C, and the Frenet curvatures of C are $|\alpha|_{|C}$, $\|\tau_j\|_{|C}$ where $2 \leqslant j \leqslant i$. This leads us to introduce the following

Def. 2. Let $i\colon M^n \to M^{n+p}$ be an isometric immersion of a n-dimensional manifold, in a $(n+p)$-dimensional manifold.

We suppose that the second fundamental form σ has the following expression

$$(*) \qquad \sigma(X, Y) = \alpha\omega(X)\omega(Y)\xi_1 \qquad \forall X, Y \in TM^n,$$

where α is a C^∞ function on M^n, ω is an unit 1-form on M^n. Then, M^n is called a strongly *cylindrical submanifold.*

Moreover, suppose that $\alpha \neq 0$, and that there exists $j-1$ non null 1-form $\tau_2, \ldots, \tau_j$, closed and proportional to ω, and j unit orthonormal vector fields in the normal bundle $\xi_1, \ldots, \xi_j$ such that System 1 is satisfied. Then M^n is called a *j-non degenerated strongly cylindrical submanifold* (s.c.-*submanifold*).

Remarks. This definition is local. Of course, a *cylinder* can be i_1-non degenerated on an open set U_1 and i_2-non degenerated on an other open set U_2, $i_1 \neq i_2$.

Using Chen's terminology, [3], condition (*) means that the direction ξ_1 is cylindrical.

Using [7], it is easy to see that if $\tilde{M}^{n+p} = E^{n+p}$, a complete strongly cylindrical submanifold which is 1-non degenerated is a cylinder.

A cone, a tangent bundle of a curve, are strongly cylindrical in E^3.

The i-th principal normal space E_i of a strongly cylindrical submanifold is $[\xi_i]$, the space spanned by the direction ξ_i.

The external curvature of a strongly cylindrical submanifold M are $k_1^{(M)} = |\alpha|$, $k_2^{(M)} = \tau_2, \ldots, k_j^{(M)} = \|\tau_j\|$.

Since $d\tau_2 = 0$, the theorem of Fröbenius implies that a 2-non degenerated strongly cylindrical submanifold of $\tilde{M}^{n+p}$ is foliated by totally geodesic $(n-1)$-dimensional submanifolds of $\tilde{M}^{n+p}$.

Now, we shall deduce from the definition and the Gauss-Codazzi equations some relations between the curvature of M^n and the curvature of $\tilde{M}^{n+p}$. We have the following

Proposition 1. *Let $i\colon M^n \to \tilde{M}^{n+p}$ an isometric immersion such that M^n*

is a strongly cylindrical submanifold of $\tilde{M}^{n+p}$.

(1) $\langle \tilde{R}(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle \quad \forall Z, Y, Z, W \in TM^n$,

(2) $\langle \tilde{R}(X, Y)\xi, \xi'\rangle = \langle R^{\perp}(X, Y)\xi, \xi'\rangle \quad \forall X, Y \in TM^n,\ \xi, \xi' \in T^{\perp}M^n$,

(3) $$\langle \tilde{R}(X, Y)Z, \xi\rangle = \big(d(\alpha\omega)(X, Y)\omega(Z) + \alpha\omega(Y)\langle Z, \nabla_X T\rangle - \alpha\omega(X)\langle Z, \nabla_Y T\rangle\big)\langle \xi_1, \xi\rangle \quad \forall X, Y, Z \in TM^n,\ \forall \xi \in T^{\perp}M^n,$$

where T *is the tangent vector field defined by* $\langle T, X\rangle = \omega(X)$.

(4) $R^{\perp}(X, Y)\xi_i = 0$.

Proof of Proposition 1. (1) Using the Gauss-Codazzi equations, we find

$$\langle \tilde{R}(X, Y)Z, W\rangle - \langle R(X, Y)Z, W\rangle = \alpha^2\omega(X)\omega(Y)\omega(Z)\omega(W) - \alpha^2\omega(Z)\omega(W)\omega(X)\omega(Y) = 0 \quad \forall X, Y, Z, W \in TM^n .$$

(2) Using the Codazzi-Ricci equations, we find

$$\langle \tilde{R}(X, Y)\xi, \xi'\rangle - \langle R^{\perp}(X, Y)\xi, \xi'\rangle = \alpha^2\omega(X)\omega(Y)\langle \xi, \xi_1\rangle\langle \xi', \xi_1\rangle - \alpha^2\omega(Y)\omega(X)\langle \xi, \xi_1\rangle\langle \xi', \xi_1\rangle = 0 \quad \forall X, Y \in TM^n,\ \forall \xi, \xi' \in T^{\perp}M^n .$$

(3) Using the Codazzi-Ricci equations, we find

$$(\tilde{R}(X, Y)Z)^{\perp} = (\overline{\nabla}_X\sigma)(Y, Z) - (\overline{\nabla}_Y\sigma)(X, Z)$$
$$= \nabla_X^{\perp}(\alpha\omega(Y)\omega(Z)\xi_1) - \nabla_Y^{\perp}(\alpha\omega(X)\omega(Z)\xi_1) - \alpha\langle \nabla_X Y, T\rangle\omega(Z)\xi_1 - \alpha\omega(Y)\langle \nabla_X Z, T\rangle\xi_1 + \alpha\langle \nabla_Y X, T\rangle\omega(Z)\xi_1 + \alpha\omega(X)\langle \nabla_Y Z, T\rangle\xi_1 ,$$

(where T is the unit tangent vectoı field defined by $\langle T, X\rangle = \omega(X)$). We obtain

$$(\tilde{R}(X, Y)Z)^{\perp} = [d(\alpha\omega)(X, Y)\cdot\omega(Z) + \alpha\omega(Y)\langle Z, \nabla_X T\rangle - \alpha\omega(X)\langle Z, \nabla_Y T\rangle]\xi_1 + [\alpha\omega(Y)\omega(Z)\tau_2(X) - \alpha\omega(X)\omega(Z)\tau_2(Y)]\xi_2$$

Since $\tau_2 \wedge \omega = 0$, the component on ξ_2 is null. The third part of the theorem follows.

(4) We have

$$\nabla^\perp_Y \nabla^\perp_X \xi_1 = \nabla^\perp_Y[\tau_2(X)\xi_2] = Y \cdot \tau_2(X)\xi_2 + \tau_2(X)[\tau_3(Y)\xi_3 - \tau_2(Y)\xi_1] .$$

Then $R^\perp(X, Y)\xi_1 = \mathrm{d}\tau_2(X, Y)\xi_2 + (\tau_2 \wedge \tau_3)(X, Y)\xi_3 = 0$ since $\mathrm{d}\tau_2 = 0$ and $\tau_2 \wedge \tau_3 = 0$.

In the general case,

$$\begin{aligned}\nabla^\perp_Y \nabla^\perp_X \xi_i &= \nabla^\perp_Y[\tau_{i+1}(X)\xi_{i+1} - \tau_i(X)\xi_{i-1}] \\ &= Y \cdot \tau_{i+1}(X)\xi_{i+1} + \tau_{i+1}(X)[\tau_{i+2}(Y)\xi_{i+2} - \tau_{i+1}(Y)\xi_i] \\ &\quad - Y \cdot \tau_i(X)\xi_{i-1} - \tau_i(X)[\tau_i(Y)[\tau_i(Y)\xi_i - \tau_{i-1}(Y)\xi_{i-2}] .\end{aligned}$$

Then

$$\begin{aligned}R^\perp(X, Y)\xi_i = {} & \mathrm{d}\tau_{i+1}(X, Y)\xi_{i+1} + (\tau_{i+1} \wedge \tau_{i+2})(X, Y)\xi_{i+2} \\ & - \mathrm{d}\tau_i(X, Y)\xi_{i-1} - (\tau_{i-1} \wedge \tau_i)(X, Y)\xi_{i-2} \\ = {} & 0 \text{ since } \mathrm{d}\tau_i = \mathrm{d}\tau_{i+1} = 0 \text{ and } \tau_{i+1} \wedge \tau_{i+2} = \tau_{i-1} \wedge \tau_i = 0 .\end{aligned}$$

Studying every case in such a way, we can conclude that $R^\perp(X, Y)\xi_i = 0$ $\forall i \in \{1, \ldots, j\}$.

2 - Cylindricity in space forms

In this paragraph, we shall give some relations between strongly cylindrical submanifolds in spaces of constant curvature and submanifolds such that the first principal normal space is of dimension 1, in spaces of constant curvature. We will prove the

Theorem 1. *Let $\tilde{M}^{n+p}$ be a manifold of constant curvature C of dimension $n + p$. Let i: $M^n \mapsto \tilde{M}^{n+p}$ an isometric immersion of a connected n-dimensional manifold M^n into $\tilde{M}^{n+p}$. We suppose that the first principal normal space E_1 satisfies the condition* $\dim E_1 \leqslant 1$. *Then,* $M^n = \overline{M}'$, *with* $M' = M_1 \cup M_2$, *where*

M_1 and M_2 are two disjoint open sets such that

each connected component of M_1 is an hypersurface of a $(n+1)$ dimensional totally geodesic submanifold of $\tilde{M}^{n+p}$,

M_2 is a strongly cylindrical submanifold of $\tilde{M}^{n+p}$.

Proof of the theorem. We need the following lemmas.

Lemma 1. *Let $i\colon M_1^n \mapsto \tilde{M}^{n+p}$ be an isometric immersion of a connected n-dimensional manifold M^n in a $n+p$ dimensional manifold of constant curvature $\tilde{M}^{n+p}$. We assume that the first principal normal space of M^n is of dimension 1 at every point, and that the second external curvature $k_2^{(M)}$ is null every where. Then, M_1^n is an hypersurface of a totally geodesic submanifold M'^{n+1} of dimension $n+1$.*

Lemma 2. *Let $i\colon M_2^n \mapsto \tilde{M}^{n+p}$ be an isometric immersion of a n-dimensional manifold M^n in a $n+p$ dimensional manifold of constants curvature $\tilde{M}^{n+p}$. We assume that the first principal normal space of M^n is of dimension 1 at every point, and that the second external curvature $k_2^{(M)}$ is never null. Then M_2^n is a strongly cylindrical submanifold of $\tilde{M}^{n+p}$ (at least 2-non degenerated).*

Proof of Lemma 1. See [**4**]$_1$ and [**8**].

Since $k_2^{(M)} = 0$, E_1 is parallel. Then, we can reduce the codimension of the immersion, and we can find a totally geodesic submanifold of dimension $n + \dim E_1$ which contain M_1^n.

Proof of Lemma 2. See [**4**]$_1$. We use the Codazzi-Ricci equations.

We can now prove the theorem. Let $p \in M^n$.

(A) Suppose that there exists an open neighborhood U_1 of p, such that $\dim E_{1|U_1} = 0$. Then, U_1 is totally geodesic, and consequently, U_1 is a strongly cylindrical submanifold (with $\alpha = 0$).

(B) Suppose that there exists an open neighborhood U_2 of p, such that $\dim E_{1|U_2} = 1$, and $k_{2|U_2}^{(M)} = 0$. Then, using Lemma 1, we can conclude that there exists an open neighborhood of p which is a hypersurface of a totally geodesic submanifold of M^{n+p}.

(C) Suppose that there exists an open neighborhood U_3 of p such that $\dim E_{1|U_3} = 1$ and $k_{2|U_3}^{(M)} \neq 0$. Then, using Lemma 2, we can conclude that U_3 is a strongly cylindrical submanifold.

It is clear that for every point of a dense open set of M^n, one of these three conditions is satisfied. The theorem follows.

Remarks. (1) In $[4]_1$ the following theorem is proved.

Theorem 2. *Let $\tilde{M}^{n+p}$ be a space form of dimension $n+p$. Let $i\colon M^n \to \tilde{M}^{n+p}$ be an isometric immersion of a n-dimensional manifold M^n. We assume that M^n is connected, simply connected, complete;* $\dim E_1 = 1$; *$k_2^{(M)} \neq 0$ at every point; there exists $j \in [1, \ldots, p]$, such that $k_j^{(M)} = cst \neq 0$.*

Then $\tilde{M}^{n+p} = E^{n+p}$, and M^n is a complete cylinder of E^{n+p} i.e. *$M^n = C \times E^{n-1}$, such that the Frenet-curvatures of C are the external curvature of M^n.*

(2) Using the Codazzi-Ricci equations, it is easy to prove that a cylinder of a manifold of constant curvature c, has a flat normal connexion and has constant curvature c.

3 - Cylindricity in a Kählerian manifold

Let $\tilde{M}^{n+p}$ be a Kähler manifold of even real dimension $n+p$, with complex structure J. We shall investigate the geometric properties of a s.c.-submanifold M^n in $\tilde{M}^{n+p}$. We begin with the following.

Remark. Let M^n be an holomorphic submanifold of a Kähler manifold $\tilde{M}^{n+p}$. If M^n is a strongly cylindrical submanifold of $\tilde{M}^{n+p}$, then M^n is totally geodesic.

In fact, M^n is minimal; this implies that $\alpha = 0$ and $\sigma = 0$.

This remark leads us to consider only the antiholomorphic, the totally real, the C.R. submanifolds which are s.c. (we shall write respectively antiholomorphic, totally real, C.R. s.c. submanifold).

(a) *The case of an antiholomorphic or totally real cylinder.*

We will prove the following

Theorem 3. (1) *There exists no antiholomorphic strongly cylindrical submanifold M^n in any positively or negatively curved Kähler manifold $\tilde{M}^{n+p}$, $p \geqslant 2$.*

(2) *There exists no totally real p-non degenerated strongly cylindrical submanifold in any positively or negatively curved Kähler manifold $\tilde{M}^{n+p}$.*

(3) *Every totally real p non degenerated strongly cylindrical submanifold M^n of any Kähler manifold $\tilde{M}^{n+p}$ is flat. (In particular, if M^n is complete, simply connected, connected; then M^n is isometric to the Euclidean space).*

Proof of Theorem 3. Let M^n be a strongly cylindrical submanifold of a Kähler manifold $\tilde{M}^{n+p}$.

(1) We have: $\langle \tilde{R}(X, Y)J\xi_1, J\eta\rangle = \langle \tilde{R}(X, Y)\xi_1, \eta\rangle$ $\forall X, Y \in TM^n$, $\forall \eta \in T^{\perp}M^n$ Since M^n is antiholomorphic, $J\xi_1$ and $J\eta$ are in TM^n.

From Proposition 1, we obtain: $\langle \tilde{R}(X, Y)J\xi_1, J\eta\rangle = \langle R(X, Y)J\xi_1, J\eta\rangle = \langle R^{\perp}(X, Y)\xi_1, \eta\rangle = 0$. Let $\eta \in T^{\perp}M^n$ such that $\xi_1 \perp \eta$. We have

$$\langle \tilde{R}(J\xi_1, J\eta)J\eta, J\xi_1\rangle = 0 .$$

This is impossible since $\tilde{M}^{n+p}$ is negatively or positively curved.

(2) We have $\langle R^{\perp}(X, Y)\xi_i, \xi_j\rangle = 0$, where $i, j \in \{1, \ldots, p\}$. $\forall X, Y \in TM^n$. This implies $\langle \tilde{R}(X, Y)\xi_i, \xi_j\rangle = 0$ from Proposition 1. Since M^n is totally real, we obtain $\langle \tilde{R}(X, Y)JZ, JW\rangle = 0$ $\forall X, Y, Z, W \in TM^n$. Consequently, $\langle \tilde{R}(X, Y)Z, W\rangle = 0$ $\forall X, Y, Z, W \in TM^n$, which is excluded, since $\tilde{M}^{n+p}$ is positively or negatively curved.

(3) Since M^n is p-nondegenerated, $R^{\perp}(X, Y) = 0$. If the cylinder is totally real, we have $\langle \tilde{R}(X, Y)JZ, JW\rangle = \langle \tilde{R}(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle = \langle R^{\perp}(X, Y)JZ, JW\rangle = 0$ $\forall X, Y, Z, W \in TM^n$. Then $R = 0$ and M^n is flat.

(b) *The case of a C.R. strongly cylindrical submanifold.*

We shall prove the following

Theorem 4. *Let M^n be a* C.R. *strongly cylindrical submanifold in a Kähler manifold $\tilde{M}^{n+p}$.*

We assume that the holomorphic distribution $\mathscr{D}$ of TM^n is involutive. Then, M^n is locally the Riemannian product $M_1 \times M_2$, where M_1 is a holomorphic totally geodesic submanifold of M^{n+p}, and M_2 is a totally real strongly cylindrical submanifold of M^{n+p}.

If M^n is p-non degenerated, M_2 is flat. If $k_2^{(M^n)} = 0$, M_2 is locally the Riemannian product $C \times \overline{M}_2$, where C is a curve in M^{n+p} with vanishing torsion, and $\overline{M}_2$ is totally geodesic in M^{n+p}. (In this case, M^n is locally the Riemannian product $C \times M'$, where C is a totally real curve, with vanishing torsion, and M' is a s.c. *submanifold).*

Before the proof of this theorem, we shall give the following

Corollary 1. *Let M^n be an antiholomorphic strongly cylindrical submanifold of a Kähler manifold. Then the following conditions are equivalent:*

(a) *The distribution $\mathscr{D}$ is integrable.*

(b) $\forall p \in M^n$, $\exists U$, *neighborhood of* p, *such that, either* U *is totally geodesic, or* $T_{|U} \in \mathscr{D}^\perp$ (T *defined by* $\langle T, X\rangle = \omega(X)$).

(c) M *is locally the product of a holomorphic submanifold and a totally real submanifold.*

Proof of Theorem 4. Let T be the unit vector field defined by $\langle T, X\rangle = \omega(X)$. We shall consider the following possibilities.

(a) $T \in \mathscr{D}^\perp$. In this case, we have

$$0 = \langle \sigma(X, Y), JZ\rangle = \langle \nabla_X Z, JY\rangle \qquad \forall X \in TM^n,\ \forall Y \in \mathscr{D},\ \forall Z \in \mathscr{D}^\perp .$$

Then, $\mathscr{D}^\perp$ is parallel. Consequently, we can write, locally $M^n = M_1 \times M_2$, where M_1 is a holomorphic and totally geodesic submanifold, integral of $\mathscr{D}$, M_2 is a totally real submanifold, integral of $\mathscr{D}^\perp$. Since $\sigma_{|\mathscr{D}^\perp}$ has the following expression

$$\sigma_{|\mathscr{D}^\perp}(Z, Z') = \alpha\omega(Z)\,\omega(Z')\,\xi_1 \qquad \forall Z, Z' \in \mathscr{D}^\perp ,$$

it is clear that M_2 is a strongly cylindrical submanifold of M^{n+p}.

(b) $\exists U$, an open set, on which $T \notin \mathscr{D}^\perp$. We have $pr_{\mathscr{D}} T \neq 0$ on U. Let $T' = pr_{\mathscr{D}} T$ on U. Since $\mathscr{D}$ is involutive, let us consider an integral submanifold M_1 of $\mathscr{D}$. Let σ' be the second fundamental form of M_1 in $\tilde{M}^{n+p}$. We have

$$\sigma'(X, Y) = \alpha\langle X, T'\rangle\langle Y, T'\rangle \xi_1 + pr_{\mathscr{D}^\perp} \nabla_X Y \qquad \forall X, Y \in \mathscr{D} .$$

M_1 is holomorphic; then, M_1 is minimal. This implies $\alpha_{|U} = 0$. Consequently, $\sigma = 0$ on U. As in (a), we can conclude that $\mathscr{D}^\perp$ is parallel, and we can write, locally $M^n = M_1 \times M_2$, where M_1 is a holomorphic and totally geodesic submanifold, integral of $\mathscr{D}$, M_2 is a totally real, and totally geodesic submanifold, integral of $\mathscr{D}^\perp$.

Proof of Corollary 1. Since M^n is antiholomorphic in $\tilde{M}^{n+p}$, $J\xi_1 \in TM^n$. Let $J\xi_1 = Z$.

Using a result of Blair and Chen [**2**], condition $\mathscr{D}$ integrable is equivalent to equation $\langle \sigma(X, JY), JZ\rangle = \langle \sigma(JX, Y), JZ\rangle$ $\forall X, Y \in \mathscr{D}$.

If $T \in \mathscr{D}^\perp$, this condition is always satisfied. If on a neighborhood U of a point $p \in M^n$, $T' = pr_{\mathscr{D}} T \neq 0$, then we have $\langle \sigma(JT', JT'), JZ\rangle = \langle \sigma(T', T'), JZ\rangle$. This implies $\alpha = 0$ and U is totally geodesic. We conclude that (a) implies (b).

Let U be an open set of M^n. If U is totally geodesic, we deduce from [2] that U is locally the product of two totally geodesic submanifolds. If $T \in \mathscr{D}^\perp$ on U, we deduce from the proof of Theorem 4 (a) that U is locally the product of two submanifolds $M_1 \times M_2$, where M_1 is holomorphic and M_2 is totally real. Therefore (b) implies (c).

Finally, (c) implies (a) is obvious.

4 - Isometric immersions of a product of two manifolds

We need the following

Def. Let $f\colon M_1 \times \ldots \times M_k \mapsto \boldsymbol{E}^N$ an isometric immersion. f is called *a product of immersions* if $f = (f_1, \ldots, f_k)$, $\boldsymbol{E}^N = \boldsymbol{E}^{m_1} \times \ldots \times \boldsymbol{E}^{m_k}$, where $f_i\colon M_i \mapsto \boldsymbol{E}^{m_i}$ is an isometric immersion.

In [6] and [1], J. D. Moore, S. Alexander and R. Maltz have studied the isometric immersion of $M = M_1 \times \ldots \times M_k$ in the Euclidean space. The y proved

Theorem A (J. D. Moore). For $1 \leqslant i \leqslant k$, let M_i be a complete connected Riemannian manifold of non negative curvature and $\dim n_i \geqslant 2$, $M = M_1 \times \ldots \times M_k$ the Riemannian product, and $\boldsymbol{E}^N$ an euclidean space of dimension $N = (\sum_1^k n_i) + k$. Then any isometric immersion $f\colon M \mapsto \boldsymbol{E}^N$ satisfies at least one of the following conditions:

(a) It is a product of hypersurface immersions.

(b) It carries a complete geodesic onto a straight line in $\boldsymbol{E}^N$.

Theorem B (S. Alexander and R. Maltz). Let $M_1, \ldots, M_k$ be complete non flat Riemannian manifolds satisfying the condition

«No M_i contains an open submanifold which is isometric to the Riemannian product $E^{n-1} \times (-\varepsilon, \varepsilon)$».

Then, any k-dimensional isometric immersion of the Riemannian product $M = M_1 \times \ldots \times M_k$ in euclidean space is a product of hypersurface immersions.

With the same technics than S. Alexander and R. Maltz, and using Theorem 1, we can prove the

Theorem 5. *Let $M_1, \ldots, M_k$ be connected, complete Riemannian manifolds, of dimensions $n_1, \ldots, n_k$. Let $f\colon M = M_1 \times \ldots \times M_k \to \boldsymbol{E}^N$ be an isometric immersions of M in euclidean space $\boldsymbol{E}^N$. Assume that the dimension of the first principal normal space $\boldsymbol{E}_1$ is k. Then, there exists a M_i which contains an open strongly cylindrical submanifold or f is a product of hypersurface immersions.*

In order to obtain a complete classification of the product of the submanifolds such that the first principal normal space has dimension two, we shall, give a description of all the immersions $f\colon M_1 \times M_2 \to E^N$ which are product of immersions, and such that $\dim E_1 = 2$.

Theorem 6. *Let M_1 and M_2 be two connected manifolds of dimension n_1 and n_2. Let $f = (f_1, f_2)\colon M = M_2 \times M_2 \to E^N$ be a product of immersions such that* $\dim E_1 = 2$. *Then, one of the two following possibilities happens*:

(a) *M_1 or M_2 is an hypersurface of an Euclidean subspace E^{n_1+1} or E^{n_2+1}.*

(b) *For $i = 1, 2$, $M_i = \overline{M}'_i$, where $\overline{M}'_i = M'_{i\alpha} \cup M'_{i\beta}$, such that $M'_{i\alpha}$ is locally an hypersurface of an Euclidean subspace E^{n_i+1}, and M'_i is a strongly cylindrical submanifold of E^N.*

Corollary 2. *Let M_1 and M_2 be two connected manifolds of dimension n_1 and n_2. Let $f = (f_1, f_2)\colon M = M_1 \times M_2 \to E^N$ be a product of immersions such that* $\dim E_1 = 2$. *Then, M_1 or M_2 is the closure of a manifold M', where $M' = M'_\alpha \cup M'_\beta$, such that each connected component of M'_α is a hypersurface of an Euclidean subspace of E^N, and M'_β is a strongly cylindrical submanifold of E^N.*

We shall use the following notations: $f = (f_1, f_2)$, $f_1\colon M_1 \to E^{N_1}$, $f_2\colon M_2 \to E^{N_2}$, $E^N = E^{N_1} \times E^{N_2}$; σ is the second fondamental form associated to f; σ_1 (resp. σ_2) is the second fondamental form associated to f_1 (resp. f_2); k_2^1 (resp. k_2^2) is the second external curvature associated to f_1 (resp. f_2), E_1^1 (resp. E_1^2) is the first principal normal space associated to f_1 (resp. f_2).

We need the following

Lemma 3. *Under the assumptions of Theorem 6, we have* $\dim E_1^1 = 2$ *and* $\dim E_1^2 = 0$ *at every point, or* $\dim E_1^1 = 0$ *and* $\dim E_1^2 = 2$ *at every point, or* $\dim E_1^1 = 1$ *and* $\dim E_1^2 = 1$ *at every point.*

Proof of Lemma 3. Let p be a point of M. Since the map $p \to \dim E^i_{1_p}$ is lower semi-continuous, and $\dim E_1 = \text{cst.} = 2$, there exists a neighborhood U of p on which $\dim E_1^1 = cst$, $\dim E_1^2 = \text{cst}$.

(a) Suppose that $\dim E_1^1 = 0$, and $\dim E_1^2 = 2$ on U. Let us consider $W_1 = \{p \in M^n / \dim (E_1^1)_p = 0\} = \{p \in M^n / \dim (E_1^2)_p > 1\}$. $W_1 \neq \emptyset$, W_1 is open and closed. Since M^n is connected, $W_1 = M^n$, and consequently, $\dim E_1^1 = 0$ on M_1.

(b) Suppose that $\dim E_1^1 = \dim E_1^2 = 1$ on U. Then, at every point, $\dim E_1^1 = \dim E_1^2 = 1$. For if there exists a point q such that $\dim E^1_{1_q} = 0$, then, $\dim E^2_{1_q} = 2$. Using (a) we obtain a contradiction.

Proof of Theorem 6 and of Corollary 2. Theorem 6 follows immediately from Lemma 3 and Theorem 1. Corollary 2 is an obvious consequence of Theorem 6.

Remark. During the print of this paper, B. Y. Chen and L. Verstraelen communicate to me that they are studying *cylindricity* in symmetric spaces. Their definition of *cylindriticity* is a little weaker than mine.

Bibliography

[1] S. Alexander and R. Maltz, *Isometric immersion of Riemannian products in euclidean space*, J. Diff. Geom. **11** (1976).

[2] D. E. Blair and B. Y. Chen, *On C.R. submanifolds of Hermitian manifolds* (to appear).

[3] B. Y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York 1973.

[4] J. Grifone and J. M. Morvan: $[\cdot]_1$ *External curvatures and internal torsion of Riemannian submanifold*, J. Diff. Geom. (to appear); $[\cdot]_2$ *Courbures de Frenet d'une sous-variété d'une variété riemannienne et « cylindricité »*, C. R. Acad. Sc. Paris (A) **283** (1976), 207-210; $[\cdot]_3$ *Courbures externes at torsion interne d'une sous-variété d'une variété riemannienne*, C. R. Acad. Sc. Paris (A) **285** (1977), 67-70; $[\cdot]_4$ *Sur les sous-variétés à torsion interne et externe constante*, C. R. Acad. Sc. Paris (A) **285** (1977), 129-133.

[5] Kon K. Yano, *Antiinvariant submanifolds*, Marcel Dekker, New York 1976.

[6] J. D. Moore, *Isometric immersions of Riemannian products*, J. Diff. Geom. (1971).

[7] O. Neill, *Isometric immersions of flat Riemannian manifolds in Euclidean space*, Michigan Mat. J. **9** (1962).

[8] M. Spivak, *Differential Geometry*, Pubblish or Perish inc., Berkeley 1979.

* * *