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Perturbations of differential systems with symmetries and the alternative method (**)

1 - Introduction

In this paper we are concerned with the existence of a family of solutions for a system of n ordinary differential equations with homogeneous linear boundary conditions which satisfy a given symmetry condition. We consider systems of the form

$$x' = A(t)x + \varepsilon f(t, x, \varepsilon) \qquad t \in [-a, a], \tag{1}$$

$$B_1 x(-a) + B_2 x(a) = 0, \tag{2}$$

where $x = \operatorname{col}(x_1, \ldots, x_n)$, ε is a small real parameter, $f = \operatorname{col}(f_1, \ldots, f_n)$, $A(t)$ is and $n \times n$ matrix, and B_1 and B_2 are constant $m \times n$ matrices.

Several authors have considered the problem of the existence of solutions for differential systems with symmetry. The problem of periodic solutions for these systems has been studied by J. K. Hale using alternative methods. Hale assumes the differential system satisfies a symmetry condition called property (E) with respect to S (see **[7]**$_3$, p. 267). A formalization of property (E) in terms of projection operators has been given by A. Stokes **[12]**. D. C. Lewis extended the concepts of periodicity and property (E) in his paper on autosynartetic solutions of differential systems **[9]**. Lienard systems with symmetries have been studied by V. E. Bononcini **[1]**$_{1,2,3}$, and T. T. Bowman **[2]**$_{1,2}$.

In a paper by D. H. Sattinger on group representation theory and nonlinear functional analysis **[11]**, the question of the bifurcation of solutions is

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studied with the symmetry assumptions given by assumptions of invariance under some transformation group.

In this paper we extend the results of Hale $[\mathbf{7}]_3$ to general homogeneous boundary conditions for systems with property (E) with respect to S. We give sufficient conditions for the existence of a family of solutions to our boundary value problem and discuss properties of these solutions. A connection with bifurcation theory is given, also in connection with the bifurcation papers of Hale $[\mathbf{7}]_{1,4}$, Cesari $[\mathbf{3}]_1$, and Gambill and Hale [**6**].

In **2** we discuss the alternative method which we will use to study our problem and give the necessary background material. In **3** we prove our main results concerning the existence of families of solutions to (1), (2). In **4** we discuss the consequences of our results and the properties of the solutions to our problem. Examples are given to illustrate that the conditions of our existence theorems can be readily verified using our knowledge of the unperturbed linear system. In Example 5 we are able to show the existence of a unique solution satisfying certain symmetry conditions.

For a discussion of alternative methods as applied to nonlinear perturbations of differential systems we refer the reader to the paper [**10**].

2 - General assumptions and background

In this section we will state our basic assumptions on our system (1), (2) and develop the necessary machinery to show the existence of a family of solutions to (1), (2). We will use the Cesari-Hale alternative method as developed in $[\mathbf{3}]_2$, $[\mathbf{7}]_3$, and [**10**]. Our notation will be the same as in [**10**]. For a general discussion of boundary value problems for linear systems we refer the reader to the book by R. H. Cole [**5**] or the exposition in $[\mathbf{3}]_2$. For a discussion of the alternative method, used also in the works of Bononcini $[\mathbf{1}]_{1,2,3}$ and Bowman $[\mathbf{2}]_{1,2}$, we refer the reader to the expositions of Cesari $[\mathbf{3}]_2$ and of Hale $[\mathbf{7}]_4$.

Throughout this paper we assume that $A(t)$ is a given $n\times n$ matrix whose entries are bounded measureable functions and B_1 and B_2 are constant $m\times n$ matrices such that the $m\times 2n$ matrix (B_1B_2) has rank m. We will take $f=\operatorname{col}(f_1, \dots, f_n)$ to be an $n\times 1$ vector function defined on $[-a, a]\times R^{2n+1}$ whose entries are measurable in t for every (x, ε) and continuous in (x, ε) for every t. Moreover, we assume that for each constant R, there exists constants M and L such that whenever $|x|$, $|y|\leqslant R$, then we have for all $t\in[-a, a]$

$$|f(t, x, \varepsilon)|\leqslant M \qquad \text{and} \qquad |f(t, x, \varepsilon)-f(t, y, \varepsilon)|\leqslant L|x-y|\,,$$

where $|\cdot|$ denotes the Euclidean norm in R^n.

Let $(AC[-a, a])^n$ denote the set of n-vector functions y whose components are absolutely continuous functions on $[-a, a]$ and such that $B_1y(-a) + B_2y(a) = 0$. For y in $(AC[-a, a])^n$ define $\|y\|_1$ by $\|y\|_1 = \sup_{-a\leq t\leq a} |y(t)| + (2a)^{-1}\int_{-a}^{a}|y'(t)|\,dt$. This defines a norm on $(AC[-a, a])^n$ and $(AC[-a, a])^n$ is a Banach space with this norm. Let $(L_1[-a, a])^n$ denote the set of equivalence classes of n-vector functions y whose components are Lebesque integrable over $[-a, a]$. For y in $(L_1[-a, a])^n$ define $\|y\|_0$ by $\|y\|_0 = (2a)^{-1}\int_{-a}^{a}|y(t)|\,dt$. This defines a norm on $(L_1[-a, a])^n$ which makes $(L_1[-a, a])^n$ a Banach space. If T is a linear operator between two Banach spaces, then denote the operator norm of T by $\|T\|$.

We associate with (1), (2) the linear problem

(3) $$x' = A(t)x \qquad t\in[-a, a]\,,$$

(4) $$B_1x(-a) + B_2x(a) = 0\,,$$

and the adjoint problem

(5) $$y' = -\tilde{A}y \qquad t\in[-a, a]\,,$$

(6) $$B_3y(-a) + B_4y(a) = 0\,,$$

where B_3 and B_4 are $(2n-m)\times n$ matrices and $\sim$ denotes the transpose. The determination of B_3 and B_4 from B_1 and B_2 can be found in [**5**], p. 141.

Let U be a $n\times p$ matrix whose p columns form a basis for the solutions to the boundary value problem (3), (4), and let V be a $q\times n$ matrix whose q rows form a basis for the solutions to the adjoint boundary value problem (5), (6). Let $c = \int_{-a}^{a}\tilde{U}(s)U(s)\,ds$ and $d = \int_{-a}^{a}V(s)\tilde{V}(s)\,ds$. The $p\times p$ matrix c and the $q\times q$ matrix d are nonsingular.

For y in $(AC[-a, a])^n$ we define the projection $\mathrm{P}\colon (AC[-a, a])^n \to (AC[-a, a])^n$ by $\mathrm{P}y(t) = U(t)c^{-1}\int_{-a}^{a}\tilde{U}(s)y(s)\,ds$. It follows from the definition of U that the range of P is all of the subspace spanned by the solutions to (3), (4).

In a similar fashion we define the projection $\mathrm{Q}\colon (L_1[-a, a])^n \to (L_1[-a, a])^n$ by $\mathrm{Q}g(t) = V(t)d^{-1}\int_{-a}^{a}V(s)g(s)\,ds$. Similarly the range of Q is the subspace spanned by the solutions to the adjoint boundary value problem (5), (6). Straightforward calculations show that P and Q are bounded linear projections in their respective spaces.

The next theorem gives the existence of a partial right inverse K of the operator $d/dt - A(t)$.

Theorem 1. *If h is in $(L_1[-a, a])^n$, then a necessary and sufficient condition that the boundary value problem*

$$x' = A(t)x + h(t)\,, \qquad B_1 x(-a) + B_2 x(a) = 0$$

has a solution is that $Qh = 0$. If $Qh = 0$, then there exists a unique solution Kh of the boundary value problem such that $PKh = 0$. Furthermore, $K(I - Q)$ is a continuous linear mapping of $(L_1[-a, a])^n$ into $(AC[-a, a])^n$.

In practice K is defined using the variation of constants formula for nonhomogeneous linear systems. The proof of Theorem 1 makes use of this definition of K and the Fredholm alternative. It is given in [**10**].

We can now employe the Cesari-Hale alternative method with our projections P and Q and the partial inverse K to split the problem (1), (2) into an equivalent system of two equations.

Theorem 2. *For each fixed ε, the boundary value problem* (1), (2) *has a solution $x(t)$ if and only if $x(t)$ satisfies the boundary condition* (2) *and the system*

$$x = Px + \varepsilon K(I - Q) f(t, x, \varepsilon)\,, \tag{7}$$

$$Qf(t, x, \varepsilon) = 0\,. \tag{8}$$

The proof of Theorem 2 in our particular case can be found in [**10**] or in a functional analysis setting in the paper by Cesari [**3**]$_2$.

Equation (7) is amenable to a fixed point method. Our next theorem will show that equation (7) has a unique solution $x^*(\alpha, \varepsilon)$ for α and ε sufficiently small and hence solving (1), (2) is reduced to finding a solution to $Qf(\cdot, x^*(\alpha, \varepsilon), \varepsilon) = 0$.

Theorem 3. *There exists $\varrho > 0$ and $\varepsilon_0 > 0$ such that, for any constant p-vector α, $|\alpha| \leqslant \varrho$ and ε such that $|\varepsilon| \leqslant \varepsilon_0$, then there exists a unique n-vector x^*, $x^* = x^*(\alpha, \varepsilon)$ such that*

$$x^* = U\alpha + \varepsilon K(I - Q)\, f(\cdot, x^*, \varepsilon)\,,$$

and x^* *satisfies the boundary conditions* (2). *Furthermore, if there is an* $\alpha = \alpha(\varepsilon)$ *with* $|\alpha(\varepsilon)| \leqslant \varrho$ *for* $|\varepsilon| \leqslant \varepsilon_0$, *such that*

$$Qf(\cdot, x^*(\alpha(\varepsilon), \varepsilon), \varepsilon) = 0 ,$$

then $x^*(\alpha(\varepsilon), \varepsilon)$ *is a solution of the boundary value problem* (1), (2).

Proof. Again the proof can be found in [**10**], but here we will outline those parts of the proof in which we have an interest.

Choose $\varrho > 0$ so that $|\alpha| \leqslant \varrho$, α a *p-vector* implies $\|U\alpha\|_1 < R$. Define $S_\alpha = \{y \text{ in } (AC[-a, a])^n : y \text{ satisfies } (2), Py = U\alpha, |y(t)| \leqslant R \text{ for all } t \in [-a, a]\}$. Let S be the union of S_α for all $|\alpha| \leqslant \varrho$. On S we define a family of maps $F(\alpha, \varepsilon)$ as follows. For $y \in S$, $F(\alpha, \varepsilon)y = U_\alpha + \varepsilon K(I - Q)f(\cdot, y, \varepsilon)$. F maps S into $(AC[-a, a])^n$. For ε sufficiently small, we can show that F is a uniform family of contractions from S into S. Hence by the Contraction Mapping Principle each $F(\alpha, \varepsilon)$ has a unique fixed point in S. In fact, the fixed point lies in S_α.

If there is an $\alpha = \alpha(\varepsilon)$ with $|\alpha(\varepsilon)| \leqslant \varrho$ for $|\varepsilon| \leqslant \varepsilon_0$, and $Qf(\cdot, x^*, (\alpha(\varepsilon), \varepsilon), \varepsilon) = 0$, then $x^*(\alpha(\varepsilon), \varepsilon)$ satisfies both equations (7), (8). Hence, it follows from Theorem 2 that x^* is a solution to (1), (2). This completes the proof of the theorem.

From the definition of Q it follows that $Qf(t, x^*, \varepsilon) = 0$ if and only if $\int_{-a}^{a} V(s)f(s, x^*(s), \varepsilon)\, ds = 0$. We now define

$$H(\alpha, \varepsilon) = \int_{-a}^{a} V(s)f(s, x^*(s), \varepsilon)\, ds . \tag{9}$$

Solving (1), (2) has been reduced to finding α and ε, $|\alpha| \leqslant \varrho$, $|\varepsilon| \leqslant \varepsilon_0$ such that

$$H(\alpha, \varepsilon) = 0 . \tag{10}$$

Equation (10) is referred to as the *bifurcation equation.*

From the definition of V it follows that $H(\alpha, \varepsilon)$ is a system of q equations where q is the number of linearly independent solutions to the adjont problem (5), (6). As defined in Theorem 3, α is a constant p-vector where p is the number of linearly independent solutions to the linear problem (3), (4). For each small ε, to solve the bifurcation equation we must solve a system of q equations in p unknowns. In the next section we will show that this can often be done under symmetry assumptions on our original system (1), (2). In this case we find that we have a family of solutions for each ε. Another approach is to use the implicit function theorem. We refer the reader to [**10**] for a discussion of systems without symmetries and their solution using the implicit function theorem.

3 - Families of solutions

In this section we will show that whenever our problem (1), (2) satisfies a symmetry condition called property (E) with respect to S, then the number of bifurcation equations is reduced. We will use this to give sufficient conditions for the existence of families of solutions to (1), (2).

A differential system $x' = g(t, x)$, where x and g are n-vectors, is said to have *property* (E) *with respect to S* if there exists a symmetric, constant, $n \times n$ matrix S such that $S^2 = I$ and $Sg(-t, Sx) = -g(t, x)$ for all $t \in [-a, a]$ and all x in R^n. If $x(t)$ is a solution to a system $x' = g(t, x)$ which has property (E) with respect to S, then $y(t) = Sx(-t)$ is also a solution for the system $x' = g(t, x)$. For

$$y' = d/dt(Sx(-t)) = -Sx'(-t) = -Sg(-t, x(-t))$$
$$= -Sg(-t, S(Sx(-t))) = g(t, Sx(-t)) = g(t, x).$$

We will need the following lemma.

Lemma 1. *Suppose* S *is a constant, symmetric,* $n \times n$ *matrix such that* $S^2 = I$. *Let* $A(t)$ *be an* $n \times n$ *matrix whose entries are bounded measurable functions over the interval* $[-a, a]$, *and let* $f \varepsilon (L_1[-a, a])^n$. *If*

(11) (a) $SA(-t) = A(t)S$, (b) $Sf(-t) = -f(t)$, (c) *if* $z(t)$ *satisfies* (2), *then* $Sz(-t)$ *also satisfies* (2), (d) *if* $z(t)$ *satisfies the adjoint boundary conditions to* (2), i.e. *equation* (6), *then so does* $Sz(-t)$,

then

(12) (a) $SU(-t)\alpha = U(t)\alpha$ *for* $t \in [-a, a]$ *if* $SU(0)\alpha = U(0)\alpha$ *for* α *a* p-*vector*, (b) $SQf(-t) = -Qf(t)$, (c) $S\{K(I-Q)f\}(-t) = \{K(I-Q)f\}(t)$.

Remark. Conditions (11a) and (11b) imply that the nonhomogeneous system $x' = A(t)x + f(t)$ has property (E) with respect to S. Assumptions (11c) and (11d) on the boundary conditions insure $Sz(-t)$ is a solution to the boundary value problem (or its adjoint) whenever z is a solution to the boundary value problem (or its adjoint).

Proof. Since A satisfies (11a) the system $x'=A(t)x$ has property (E) with respect to S. Hence $SU(-t)\alpha$ is a solution to $x'=A(t)x$ as is $U(t)\alpha$. If $SU(0)\alpha = U(0)\alpha$, then by uniqueness we must have $SU(-t)\alpha = U(t)\alpha$ for $t \in [-a, a]$. This proves (12a).

The columns of the matrix $U(t)$ form a basis for the solutions to $x'=A(t)x$ which satisfy (2). The columns of $SU(-t)$ also form a basis for the solutions to $x'=A(t)x$ which satisfy (2) since (11c) insures the columns of $SA(-t)$ satisfy (2). Therefore, there is a nonsingular $p\times p$ matrix G^0 such that $SU(-t) = U(t)G^0$, $t \in [-a, a]$. Similarly, there is a nonsingular $q\times q$ matrix G_0 such that $V(-t)S = G_0 V(t)$, $t \in [-a, a]$. Let f satisfy (11b). Then since S is selfadjoint,

$$SQf(-t) = S\tilde{V}(-t)d^{-1}\int_{-a}^{a} V(u)f(u)\,du = -\tilde{V}(t)G_0 d^{-1}G_0\int_{-a}^{a} V(u)f(u)\,du\,.$$

It follows from the definition of d that $G_0 d\tilde{G}_0 = d$. Moreover, $V(-t)S = G_0V(t)$ thus $V(-t) = V(-t)S^2 = G_0V(t)S + G_0^2V(-t)$ and $G_0^2 = I$. It now follows that $\tilde{G}_0 d^{-1}G_0 = d^{-1}$. Continuing our calculation of $SQf(-t)$ we find $SQf(-t) = Qf(t)$. This proves (12b).

It follows from (11b) and (12b) that $S(I-Q)f(-t) = (I-Q)f(t)$. Hence to prove (12c), it suffices to prove (12c) with $(I-Q)f$ replaced by f with $Qf = 0$

Let $Qf = 0$, then Kf is the unique solution to the boundary value problem

$$x'=A(t)x + f(t) \qquad t\in[-a, a]\,, \tag{13}$$

$$B_1x(-a) + B_2x(a) = 0\,, \tag{14}$$

with $Px = 0$.

Assumptions (11a) and (11b) imply that system (13) has property (E) with respect to S; hence $SKf(-t)$ is also a solution to (13). Since Kf satisfies (14), (11c) implies $SKf(-t)$ also satisfies (14).

Since Kf is a unique solution of (13), (14) with $PKf = 0$, it remains only to show that $P(SKf)(-\cdot) = 0$. Now

$$P\{(SKf)(-\cdot)\}(t) = U(t)c^{-1}\int_{-a}^{a}\tilde{U}(u)SKf(-u)\,du = U(t)c^{-1}\int_{-a}^{a}\tilde{G}^0\tilde{U}(-u)Kf(-u)\,du$$

$$= U(t)c^{-1}\tilde{G}^0\{\int_{-a}^{a}\tilde{U}(u)Kf(u)\,du\} = 0\,.$$

Hence $SKf(-\cdot) = Kf(\cdot)$. This proves (12c) and completes the proof of Lemma 1.

The next theorem gives conditions for the linear dependence of the bifurcation equations.

Theorem 4. *Let $x^*(\alpha, \varepsilon)$ be the function which is the solution to the auxiliary equation corresponding to a particular α and ε. Suppose that system* (1) *has property* (E) *with respect to S all $|\varepsilon| \leqslant \varepsilon_0$, and that assumptions* (11c) *and* (11d) *of Lemma* 1 *are satisfied. Then*

$$\text{(15)} \qquad \mathrm{S}x^*(\alpha, \varepsilon)(-t) = x^*(\alpha, \varepsilon)(t)\,,$$

for all $t \in [-a, a]$ provided $SU(0)\alpha = U(0)\alpha$. Moreover, if $H(\alpha, \varepsilon)$ denotes the vector defined by equation (9), *then*

$$\text{(16)} \qquad (\mathrm{I} + \mathrm{G}_0)H(\alpha, \varepsilon) = 0\,,$$

where G_0 is the nonsingular $q \times q$ matrix such that $V(-t)\mathrm{S} = \mathrm{G}_0 V(t)$ for $t \in [-a, a]$.

Proof. Let $\mathrm{S}U(0)\alpha = U(0)\alpha$ and take S_α as defined in the proof of Theorem 3. Let S^*_α be the subset of S made up of those y in S_α such that $\mathrm{S}y(-t) = y(t)$. Recall $F(\alpha, \varepsilon)\colon \mathrm{S}_\alpha \to \mathrm{S}_\alpha$ for $|\alpha| \leqslant \varrho$, $|\varepsilon| \leqslant \varepsilon_0$ was given by $F(\alpha, \varepsilon) = U_\alpha + \varepsilon \mathrm{K}(\mathrm{I} - \mathrm{Q})f(\cdot, y, \varepsilon)$. Since (1) has property (E) with respect to S, then for fixed y in S^*_α, $x' = A(t)x + \varepsilon(t, y(t), \varepsilon)$ has property (E) with respect to S $\big($note: $\mathrm{S}F(-t, y(-t), \varepsilon) = \mathrm{S}f(-t, \mathrm{S}y(t), \varepsilon) = -f(t, y(t), \varepsilon)\big)$ hence $\varepsilon\mathrm{K}(\mathrm{I} - \mathrm{Q})\, f(\cdot, y, \varepsilon)$ is in S^*_α. Moreover, if $\mathrm{S}U(0)\alpha = U(0)\alpha$, then $U(t)\alpha$ is in S^*_α hence $F(\alpha, \varepsilon)y$ is in S^*_α. $F(\alpha, \varepsilon)\colon \mathrm{S}^*_\alpha \to \mathrm{S}_\alpha$, hence the fixed point $x^*(\alpha, \varepsilon)$ of $F(\alpha, \varepsilon)$ must lie in S^*_α. That is, $\mathrm{S}x^*(\alpha, \varepsilon)(-t) = x^*(\alpha, \varepsilon)(t)$.

Using (15) and the fact $V(-t)\mathrm{S} = \mathrm{G}_0 V(t)$ we have

$$H(\alpha, \varepsilon) = \int_{-a}^{a} V(u) f(u, \mathrm{S}x^*(-u), \varepsilon)\, du = -\mathrm{G}_0 \int_{-a}^{a} V(u) f(u, x^*(u), \varepsilon)\, du = -\mathrm{G}_0 H(\alpha, \varepsilon)\,.$$

This completes the proof of Theorem 4.

Theorem 5. *Let system* (1) *have property* (E) *with respect to* S *for all $|\varepsilon| \leqslant \varepsilon_0$ and $\mathrm{S}U(0)\alpha = U(0)\alpha$ for all p-vectors α, $|\alpha| \leqslant \varrho$ where ε_0 and ϱ are given in Theorem* 3. *If* $(\mathrm{I} + \mathrm{G}_0)$ *is nonsingular, then* (1), (2) *has a p-parameter family of solutions for each ε, $|\varepsilon| \leqslant \varepsilon_0$ where the parameters are given by the components of the p-vector α.*

Proof. If $I + G_0$ is nonsingular, then $H(\alpha, \varepsilon) = 0$ for all $|\alpha| \leqslant \varrho$ and $|\varepsilon| \leqslant \varepsilon_0$ and the theorem follows from Theorem 3.

If $I + G_0$ is singular, then the q equations of $H(\alpha, \varepsilon)$ are linearly dependent. pet $G(\alpha, \varepsilon)$ denote a subset of the equations of $H(\alpha, \varepsilon)$ which are linearly inde-Lendent and span the equations of $H(\alpha, \varepsilon)$. $G(\alpha, \varepsilon) = 0$ if and only if $H(\alpha, \varepsilon) = 0$. For each ε, $G(\alpha, \varepsilon)$ is a set of r equations in p unknowns where $r < q$.

Theorem 6. *Let system* (1) *have property* (E) *with respect to* S *for all* $|\varepsilon| \leqslant \varepsilon_0$ *and* $SU(0)\alpha = U(0)\alpha$ *for all* $|\alpha| \leqslant \varrho$. *Let* $I + G_0$ *be singular and* $G(\alpha, \varepsilon)$ *and* r *as defined above. Assume* $p \geqslant r$ *and* $G(\alpha, \varepsilon)$ *is continuously differentiable with respect to both* α *and* ε *for all* $|\alpha| \leqslant \varrho$ *and* $|\varepsilon| \leqslant \varepsilon_0$. *If there is a* p*-vector* $\bar{\alpha}$ *such that* $|\bar{\alpha}| \leqslant \varrho$, $G(\bar{\alpha}, 0) = 0$, *and* $\partial G(\bar{\alpha}, 0)/\partial\alpha$ *has rank* r, *then there is an* $\varepsilon_1 > 0$ *and a solution* $x^*(\alpha(\varepsilon), \varepsilon)$, $|\varepsilon| \leqslant \varepsilon_1$ *of the problem* (1), (2). *Moreover, if* $p > r$, *then we have a* $p - r$ *parameter family of solutions of* (1), (2).

Proof. The hypotheses on $G(\alpha, \varepsilon)$ and the implicit function theorem imply there is an ε_1, $0 \leqslant \varepsilon_1 \leqslant \varepsilon_0$, such that the equation $G(\alpha, \varepsilon) = 0$ has a solution $\alpha(\varepsilon)$, $|\alpha(\varepsilon) \leqslant \varrho$, for all ε, $|\varepsilon| \leqslant \varepsilon_1$. But this implies $x^*(\alpha(\varepsilon), \varepsilon)$ is a solution to (1), (2). If $p > r$, then it also follows from the implicit function theorem that there is a $p - r$ parameter family of $\alpha(\varepsilon)$'s which satisfy $G(\alpha(\varepsilon), \varepsilon) = 0$. This completes the proof of the theorem.

The continuous differentiability of $G(\alpha, \varepsilon)$ depends upon the differentiability of the point x^* of the map $F(\alpha, \varepsilon)$ defined in the proof of Theorem 3. A discussion of the differentiability properties of x^* is given in [**10**].

4 - Discussion and examples

Under our assumptions on $A(t)$ and f if follows from Theorem 3 that for each ε, $|\varepsilon| \leqslant \varepsilon_0$, if $Qf = 0$, then we get a unique solution for each p-vector α, $|\alpha| \leqslant \varrho$. For $m = n$, if the linear problem (3), (4) has only the trivial solution and hence $P = 0$, then $Q = 0$ (see [**2**]$_2$, p. 143). Consequently, a family of solutions is likely to occur only when the linear system has nontrivial solutions. This is called the resonance case. This suggests the use of alternative methods to approach the problem.

Bifurcation theory deals with the solution of a system which depends upon a parameter. It is concerned with the existence of solutions, number of solutions, and the behaviour of these solutions. It follows from Theorems 5 and 6 that we have solutions to our system (1), (2) depending upon ε and $p - r$ other parameters. Consequently for each ε and p-vector α, there is a $p - r$ family of solutions branching out from our solution $x^*(\alpha, \varepsilon)$. For a more detailed discussion of bifurcation theory for differential systems we refer to $[\mathbf{4}]_{1,2}$, [**8**], and to $[\mathbf{3}]_1$, [**6**], $[\mathbf{7}]_{1,4}$.

When the conditions of Theorem 5 or 6 are satisfied, we obtain a family of solutions to (1), (2). However, it follows from Theorem 3 and the proof of Theorem 3 that we do get a unique solution for each p-vector α, $|\alpha| \leqslant \varrho$. It thus follows as a consequence of the Contraction Mapping Principle that our solution x^* has the form $x^*(\alpha, \varepsilon) = U(t)\alpha + O(\varepsilon)$. This representation may be used in determining the particular solution we may want (see Example 5).

If the system (1), (2) has a solution with small norm, then it follows from Theorem 3 that the solution must lie in the set S as defined in the proof of Theorem 3. If our system (1) has property (E) with respect to S, satisfies (11c), (11d), and $\mathrm{S}U(0)\alpha = U(0)\alpha$ for all p-vectors α, $|\alpha| \leqslant \varrho$, then the solution $x^*(t)$ must lie in S*. This is evident from the proof of Theorem 4 where the mapping $F(\alpha, \varepsilon)$ from S into S is actually a map from S* into S* and hence the fixed point lies in S*. It follows from the definition of S* that the solution x^* satisfies the symmetry condition $\mathrm{S}x^*(-t) = x^*(t)$. Thus in order for a small solution x^* to exist which does not have this symmetry property we must have some p-vector α such that $\mathrm{S}U(0)\alpha \neq U(0)\alpha$. Thus it becomes a property of the linear problem as to whether we get nonsymmetric solutions of small norm. This condition is easily checked as seen in Example 5 where we obtain a unique solution satisfying certain symmetry conditions.

Example 1.

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \varepsilon f(t, x) \qquad t \in [-\pi, \pi]\,, \tag{17}$$

$$\operatorname{diag}(1, 1, 1)\, x(-\pi) + \operatorname{diag}(-1, 1, -1) x(\pi) = 0\,. \tag{18}$$

We assume f satisfies the assumptions made in **2** and has the following symmetry properties:

$$f_1(-t, x_1, -x_2, x_3) = -f_1(t, x)\,, \tag{19$_1$}$$

$$f_2(-t, x_1, -x_2, x_3) = f_2(t, x)\,, \tag{19$_2$}$$

$$f_3(-t, x_1, -x_2, x_3) = -f_3(t, x)\,. \tag{19$_3$}$$

Here, $p = 2$ and $q = 2$.

Let $S = \operatorname{diag}(1, -1, 1)$, then S is a 3×3 constant symmetric matrix and $S^2 = I$. We will show (17) has property (E) with respect to S. We must show $S\{A(-t)Sx + \varepsilon f(-t, Sx)\} = -\{A(t)x + \varepsilon f(t, x)\}$.

$$SA(-t)S = \operatorname{diag}(1,-1,1)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\operatorname{diag}(1,-1,1)$$

$$= \operatorname{diag}(1,-1,1)\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -A(t)\,.$$

Now $Sx = \operatorname{diag}(1, -1, 1)x = \operatorname{col}(x_1, -x_2, x_3)$, so we have

$$Sf(-t, Sx) = \operatorname{diag}(1,-1,1)\begin{pmatrix} f_1(-t, x_1, -x_2, x_3) \\ f_2(-t, x_1, -x_2, x_3) \\ f_3(-t, x_1, -x_2, x_3) \end{pmatrix} = -f(t, x)\,.$$

It easily follows that (17) has property (E) with respect to S.

We will now show that the boundary condition (18) is such that (11c) and (11d) hold. Assume z satisfies (18), then $\operatorname{diag}(1, 1, 1)Sz(-(-1\pi)) + \operatorname{diag}(-1, 1, -1)Sz(-\pi) = -\{\operatorname{diag}(1, 1, 1)z(-\pi) + \operatorname{diag}(-1, 1, -1)z(\pi)\}$. But the term in brackets is zero since z satisfies (18). Hence $Sz(-\cdot)$ satisfies (18) also. This shows (11c) holds. The adjoint boundary conditions are the same as (18), hence (11d) also holds.

The matrices U and V are given by

$$U(t) = \begin{pmatrix} \cos t & 0 \\ -\sin t & 0 \\ 0 & 1 \end{pmatrix}, \qquad V(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now calculate G_0. Recall G_0 is the nonsingular constant 2×2 matrix such that $V(-t)S = G_0 V(t)$.

$$V(-t)S = \begin{pmatrix} \cos(-t) & -\sin(-t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\operatorname{diag}(1,-1,1)$$

$$= \begin{pmatrix} \cos t & \sin t & 0 \\ 0 & 0 & 1 \end{pmatrix}\operatorname{diag}(1,-1,1) = V(t)\,.$$

Hence $G_0 = \operatorname{diag}(1, 1)$.

Let α be any 2-vector, then

$$SU(0)\alpha = \operatorname{diag}(1,-1,1)\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}\alpha = U(0)\alpha\,.$$

Hence for any 2-vector α, $SU(0)\alpha = U(0)\alpha$. It now follows as a consequence of Theorem 5 that for $|\alpha|\leqslant\varrho$ and $|\varepsilon|\leqslant\varepsilon_0$, then we have a 2-parameter family of solutions where the two parameters are α_1 and α_2.

Example 2.

$$x' = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}x + \varepsilon f(t,x) \qquad t\in[-a,a]\,, \tag{20}$$

$$\operatorname{diag}(1,1,1)x(-a) + \operatorname{diag}(-1,1,-1)x(a) = 0\,. \tag{21}$$

Here $p = 2$ and $q = 2$. We assume f satisfies the assumptions made in **2** and the symmetry condition (19).

Let $S = \operatorname{diag}(1,-1,1)$. One can show that (20) has property (E) with respect to S and the boundary conditions (21) satisfy (11c) and (11d). For this system we have

$$U(t) = \begin{pmatrix}1 & t^2\\ 0 & 2t\\ 0 & 2\end{pmatrix}, \qquad V(t) = \begin{pmatrix}0 & 0 & 1\\ 2 & -2t & t^2\end{pmatrix}.$$

For every 2-vector α we have that $SU(0)\alpha = U(0)\alpha$. Now $G_0 = I$ and it follows from Theorem 5 that for each $|\varepsilon|\leqslant\varepsilon_0$ and $|\alpha| < \varrho$ we have a 2-parameter family of solutions where the two parameters are α_1 and α_2.

Remark. The function $f = \operatorname{col}(f_1, f_2, f_3)$ given by $f_1(t, x_1, x_2, x_3) = t^2x_1x_2 + \sin t + t^3x_3^2$, $f_2(t, x_1, x_2, x_3) = x_1^2 + x_2^2 + \cos t\, x_3^3 + x_1x_3$, $f_3(t, x_1, x_2, x_3) = x_1x_2x_3$, satisfies the symmetry conditions (19) and our assumptions in **2**.

Example 3.

$$x' = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & \sin t\end{pmatrix}x + \varepsilon f(t,x) \qquad t\in[-\pi,\pi]\,, \tag{22}$$

$$\operatorname{diag}(1,1,1)x(-\pi) + \operatorname{diag}(-1,-1,-1)x(\pi) = 0\,. \tag{23}$$

Let f satisfy our assumptions made in **2** and the symmetry condition (19). A simple calculation shows that (22) has property (E) with respect to S where $S = \operatorname{diag}(1, -1, 1)$. Moreover our boundary condition satisfies (11c) and (11d). Here we have

$$U(t) = \begin{pmatrix} \sin t & \cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & e^{-\cos t} \end{pmatrix}, \qquad V(t) = \begin{pmatrix} \sin t & \cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & e^{\cos t} \end{pmatrix}$$

and $G_0 = \operatorname{diag}(-1, 1, 1)$.

Let α be a 3-vector, then

$$S\,U(0)\,\alpha = \operatorname{diag}(1, -1, 1) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ e^{-1}\alpha_3 \end{pmatrix},$$

and

$$U(0)\,\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ e^{-1}\alpha_3 \end{pmatrix}.$$

These are equal if and only if $\alpha_1 = 0$.

Consider $\beta = \operatorname{col}(\beta_1, \beta_2) = \operatorname{col}(\alpha_2, \alpha_3)$. For α such that $\alpha_1 = 0$, we have by Theorem 4 that $\operatorname{diag}(0, 2, 2)H(\alpha, \varepsilon) = 0$. Hence $H_2 = H_3 = 0$ so $H(\alpha, \varepsilon)$ reduces to $G(\beta, \varepsilon) = H_1(\alpha, \varepsilon)$. Now

$$H_1(\alpha, \varepsilon) = \int_{-\pi}^{\pi} \sin t f_1(t, x)\, dt + \int_{-\pi}^{\pi} \cos t f_2(t, x)\, dt .$$

Hence we have

$$G(\beta, \varepsilon) = \int_{-\pi}^{\pi} \{\sin t f_1(t, x^*(\beta, \varepsilon)) + \cos t f_2(t, x^*(\beta, \varepsilon))\}\, dt .$$

We know that $x^*(\alpha, \varepsilon) = U(t)\alpha + 0(\varepsilon)$ so $x_1^*(\beta, \varepsilon) = \beta_1 \cos t + 0(\varepsilon)$, $x_2^*(\beta, \varepsilon) = -\beta_1 \sin t + 0(\varepsilon)$, $x_3^*(\beta, \varepsilon) = \beta_2 e^{-\cos t} + 0(\varepsilon)$. So for $\varepsilon = 0$, we have

$$G(\beta, 0) = \int_{-\pi}^{\pi} \{\sin t f_1(t, \beta_1 \cos t, -\beta_1 \sin t, \beta_2 e^{-\cos t}) + \cos t f_2(t, \beta_1 \cos t, -\beta_1 \sin t, \beta_2 e^{-\cos t})\}\, dt .$$

Let us consider the case where

$$f_1(t, x) = x_1 x_2 + \sin t , \qquad f_2(t, x) = x_1 x_3 .$$

Now

$$G(\beta, 0) = \int_{-\pi}^{\pi} \{\beta_1^2 \cos t \sin^2 t + \sin^2 t + \beta_1\beta_2 \cos^2 t\, e^{-\cos t}\}\, dt = \beta_1^2 A + B + \beta_1\beta_2 C\,,$$

where A, B, and C are constant determined by the above equation. Also, $\partial G(\beta, 0)/\partial\beta = (2\beta_1 A + \beta_2 C, \beta_1 C)$. Fix $\bar\beta_1 \neq 0$ and choose $\bar\beta_2 = (-\bar\beta_1^2 A - B)/\bar\beta_1 C = \varphi(\bar\beta_1)$. Then $G(\bar\beta, 0) = 0$ and the rank of $\partial G(\bar\beta, 0)/\partial\beta = 1$. So by Theorem 6 we have a one-parameter family of solutions to (22), (23) where $\bar\beta_1$ is our parameter. Hence our solution has the form $x_1^* = \bar\beta_1 \cos t + O(\varepsilon)$, $x_2^* = -\bar\beta_1 \sin t + O(\varepsilon)$, $x_3^* = \varphi(\bar\beta_1)e^{-\cos t} + O(\varepsilon)$.

Example 4.

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \varepsilon f(t, x) \qquad t \in [-\pi, \pi]\,, \tag{24}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(-\pi) + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} x(\pi) = 0\,. \tag{25}$$

Let f satisfy our assumptions made in **2** and the symmetry condition (19). Again, a simple calculation shows that (24) has property (E) with respect to S where $S = \mathrm{diag}\,(1, -1, 1)$. Our boundary conditions require $x_2(t)$ to be zero at π and $-\pi$ and $x_3(t)$ periodic with period 2π. The adjoint boundary matrices B_3 and B_4 in equation (6) are

$$B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(Refer to [**5**], p. 141, for the determination of B_3 and B_4 from B_1 and B_2.) Once B_3 and B_4 have been determined, it is easy to show that conditions (11c) and (11d) are satisfied. We have

$$U(t) = \begin{pmatrix} \cos t & 0 \\ -\sin t & 0 \\ 0 & 1 \end{pmatrix}, \qquad V(t) = \begin{pmatrix} \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $G_0 = \mathrm{diag}\,(-1, 1)$.

Let α be any 2-vector, then

$$SU(0)\alpha = \operatorname{diag}(1,-1,1)\,,\quad \begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}\alpha_1\\ 0\\ \alpha_2\end{pmatrix} = U(0)\alpha\,.$$

It follows by Theorem 4 that for any 2-vector α, $(\mathrm{I}+\mathrm{G}_0)H(\alpha,\varepsilon)=0$. Since $\mathrm{I}+\mathrm{G}_0=\operatorname{diag}(0,1)$, solving $H(\alpha,\varepsilon)=0$ reduces to solving $H_1(\alpha,\varepsilon)=0$. Now

$$H_1(\alpha,\varepsilon)=\int_{-\pi}^{\pi}\sin t f_1\big(t, x^*(\alpha,\varepsilon)(t)\big)\,dt+\int_{-\pi}^{\pi}\cos t f_2\big(t, x^*(\alpha,\varepsilon)(t)\big)\,dt\,.$$

Since $x^*(\alpha,\varepsilon)=U(t)\alpha+O(\varepsilon)$,

$$x_1^*(\alpha,\varepsilon)=\alpha_1\cos t+O(\varepsilon)\,,\quad x_2^*(\alpha,\varepsilon)=-\alpha_1\sin t+O(\varepsilon)\,,\quad x_3^*(\alpha,\varepsilon)=\alpha_2+O(\varepsilon)\,.$$

Let us consider the particular case where

$$f_1(t,x)=\sin t x_3^2+x_1x_2x_3\,,\qquad f_2(t,x)=(1+x_2^2)^{-1}-\cos t\sin x_3$$

and f_3 need only satisfy assumptions in **2** and condition (19). Then

$$H_1(\alpha,0)=\int_{-\pi}^{\pi}\alpha_2^2\sin^2 t\,dt-\int_{-\pi}^{\pi}\alpha_1^2\alpha_2\cos t\sin^2 t\,dt$$

$$+\int_{-\pi}^{\pi}\cos t(1+\alpha_1^2\sin^2 t)^{-1}\,dt-\sin\alpha_2\int_{-\pi}^{\pi}\cos^2 t\,dt=\pi\alpha_2^2-\pi\sin\alpha_2\,,$$

and

$$\partial H_1(\alpha,0)/\partial\alpha=(0\quad 2\pi\alpha_2-\pi\cos\alpha_2).$$

For $\alpha_2=0$, $H_1(\alpha,0)=0$ and $\partial H_1(\alpha,0)/\partial\alpha=(0\ \ -\pi)$. So by Theorem 6 we have a one-parameter family of solutions to (24), (25) where α_1 is our parameter. Our solution has the form $x_1^*=\alpha_1\cos t+O(\varepsilon)$, $x_2^*=-\alpha_1\sin t+O(\varepsilon)$, $x_3^*=O(\varepsilon)$.

Example 5. We seek T-periodic solutions to

$$y''=\varepsilon f(t,y)\,,\qquad t\in[-T/2,\,T/2]\,, \tag{26}$$

where $f(t,y)$ is periodic of period T in t and satisfies the assumptions of **2** and the symmetry condition $f(-t, y) = -f(t, y)$. Expressed as a first order system,

$$x' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \varepsilon g(t, x) , \tag{27}$$

$$\operatorname{diag}(1, 1)x(-T/2) + \operatorname{diag}(-1, -1)x(T/2) = 0 , \tag{28}$$

where $g(t, x) = \operatorname{col}\big(0, f(t, x_1)\big)$. The system (27) has property (E) with respect to S where $S = \operatorname{diag}(-1, 1)$. The boundary conditions (28) are the same as the adjoint boundary conditions and satisfy (11c) and (11d). We have $U(t) = \operatorname{col}(1, 0)$, $V(t) = (0, 1)$, and $G_0 = 1$. Let α be any real number, then $SU(0)\alpha = \operatorname{diag}(-1, 1)\begin{pmatrix}1\\0\end{pmatrix}\alpha = \begin{pmatrix}-\alpha\\0\end{pmatrix}$. Now $U(0)\alpha = \begin{pmatrix}\alpha\\0\end{pmatrix}$ so we have $SU(0)\alpha = U(0)\alpha$ only when $\alpha = 0$. By Theorem 4 when $\alpha = 0$, $(I + G_0)H(\alpha, \varepsilon) = 0$. Since $G_0 = 1$, we have $H(0, \varepsilon) = 0$ for ε sufficiently small.

It also follows from Theorem 4 that $Sx^*(0, \varepsilon)(-t) = x^*(0, \varepsilon)(t)$, which says that $x_1^*(0, \varepsilon)$ is an odd function. Thus equation (26) has an odd T-periodic solution for ε sufficiently small.

In Theorem 3 we find we can express $x^*(\alpha, \varepsilon) = U(t)\alpha + \varepsilon K(I - Q)f(\cdot, x^*, \varepsilon)$ where $x^*(\alpha, \varepsilon)$ is the unique solution to equation (7) with $|\alpha| \leqslant \varrho$ and $|\varepsilon| \leqslant \varepsilon_0$. If we have any solution $\tilde{x}$ to equation (1) with $P\tilde{x} = U(t)\alpha$, $|\alpha| \leqslant \varrho$, then $\tilde{x}$ must also satisfy equation (7). Since x^* was the unique fixed point in the set S_α (see proof of Theorem 3), then we must have $x^*(\alpha, \varepsilon) = \tilde{x}$.

In the particular case of the problem (27), (28), we have $x^*(\alpha, \varepsilon) = \begin{pmatrix}\alpha\\0\end{pmatrix} + \varepsilon K(I - Q)g(\cdot, x^*, \varepsilon)$, where $PK(I - P)g = 0$ or $K(I - P)g$ has mean value zero.

Consequently, $x_1^*(\alpha, \varepsilon)$ will be an odd function only if $\alpha = 0$. It thus follows that $x_1^*(0, \varepsilon)$ will be the only odd T-periodic solution to equation (26) for ε sufficiently small.

Of particular interest is the problem of T-periodic solutions to

$$y'' + \varepsilon \sin y = \varepsilon f(t) , \tag{29}$$

where f is an odd T-periodic function. The above analysis shows that equation (29) has a unique odd T-periodic solution for small α and ε. This is consistant with the results of Bononcini $[1]_{1,2}$. He found equation (29) to have at least one odd T-periodic solution and points out there are good reasons why the problem has only one such solution. While Bononcini's existence results are valid for $\varepsilon = 1$, we have been able to prove existence and uniqueness for ε small.

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Abstract

In this paper we are concerned with the existence of a family of solution for a differential system with symmetries. We use the Cesari-Hale alternative method and an extension to general homogeneous boundary conditions of a symmetry property due to J. K. Hale. We give sufficient conditions for the existence of a family of solutions and discuss the properties of these solutions.

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