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JOHN J. SACCOMAN (*)

Some results on Hahn-Banach extension properties (**)

1 - Introduction

One of the many ramifications of the Hahn-Banach theorem involves the extension of linear mappings. In this paper we consider several relationships between the Hahn-Banach sublinear extension property and the Hahn-Banach norm extension property. J. Kelley [**3**] showed that each real Banach space X with the Hahn-Banach norm extension property is isometrically isomorphic to $C(S)$, the space of continuous real valued functions on an extremally disconnected compact Hausdorff Space S. We observe that relative to the usual function ordering, $C(S)$ is an « order complete real ordered linear space with an Archimedean unit ». In this paper all linear spaces are assumed to be real.

As we indicate in the sequel, there are many interesting examples of order complete ordered linear spaces with an Archimedean unit. In this note we show that on each order complete ordered linear space X with an Archimedean unit, a norm can be constructed relative to which X has the Hahn-Banach norm extension property.

We also obtain the following interesting corollaries.

Corollary 1. *Each Banach space X with the Hahn-Banach norm extension property has a « natural » ordering with an Archimedean unit relative to which it has the Hahn-Banach sublinear extension property, and conversely, each ordered linear space which has an Archimedean unit and the Hahn-Banach sublinear extension property, has a natural norm relative to which it has the Hahn-Banach norm extension property.*

(*) Indirizzo: Dept. of Math., Seton Hall University, South Orange, NJ 07079, U.S.A.
(**) R cevuto: 10-X-1979.

Corollary 2. *Let X be a Banach space with dimension d (not nececessarily finite). Let the unit ball B in X^* (the dual space of X) be topologized with the weak star topology. If each subspace of $C(B)$ with dimension d has a nonempty intersection with the cone of positive functions in $C(B)$, then X has the Hahn-Banach norm extension property.*

2 - Preliminaries

As usual an ordered linear space (o.l.s.) X is one with a partial ordering $\leqslant$ on X relative to which translation by members of X and multiplication by positive numbers preserves order and multiplication by negative numbers reverses order. A wedge $W \subset X$ is a convex set such that $tW \subset W$ for every $t \geqslant 0$. A cone is a wedge containing no line through 0. An o.l.s. is order complete if each nonempty subset with an upper bound has a least upper bound.

Definition 2.1. A normed linear space (n.l.s.) X has the Hahn-Banach norm extension property if for any n.l.s. Y, any subspace M of Y, and any bounded linear operator (b.l.o.) T mapping M into X, there is a b.l.o. T' mapping Y into X with $\|T'\| = \|T\|$ and $T'(y) = T(y)$ for every $y \in M$.

Definition 2.2. An o.l.s. X is said to have the Hahn-Banach sublinear extension property if for any linear space Y, any subspace M of Y, any linear function f mapping M into X, and any sublinear function p (i.e., $p(y_1 + y_2) \leqslant p(y_1) + p(y_2)$ and $p(ay) = ap(y)$ for all $a \geqslant 0$), where p maps Y into X with $f(y) \leqslant p(y)$ for all $y \in M$, there is a linear function F mapping Y into X with $F(y) \leqslant p(y)$ for all $y \in Y$ and $F(y) = f(y)$ for all $y \in M$.

Definition 2.3. An Archimedean unit in an o.l.s. X is an element $e \in X$ such that $e > 0$ and for each $x \in X$ there is a positive real number t_x with $x < t_x e$.

Following the usual notation $[-e, e] = \{z \in X \mid -e \leqslant z \leqslant e\}$. Also, for any Archimedean unit e, $[-e, e]$ is radial, i.e. for each finite set $E \subset X$, there exists a real number λ_0 such that $E \subset [-\lambda e, \lambda e]$ whenever $|\lambda| \geqslant |\lambda_0|$.

3 - Examples

The following are some examples of order complete o.l.s. with an Archimedean unit.

Example 3.1. Let X be an n-dimensional real vector space with basis $\{x_1, x_2, \ldots, x_n\}$. Any $x, y \in X$ are given uniquely by

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n , \qquad y = \beta_1 x_1 + \ldots + \beta_n x_n .$$

Define the ordering $\leqslant$ by $x \leqslant y$ if $\alpha_i \leqslant \beta_i$ for $i = 1, 2, \ldots, n$. Further, define

$$x \vee y = (\alpha_1 \vee \beta_1) x_1 + \ldots + (\alpha_n \vee \beta_n) x ,$$

where $(\alpha_i \vee \beta_i) = \max\{\alpha_i, \beta_i\}$, and let $e = x_1 + \ldots + x_n$.

Example 3.2. Let X be the space of continuous real valued functions on a topological space T. Define the ordering $\leqslant$ as usual, i.e., $f \leqslant g$ if $f(x) \leqslant g(x)$ for each $x \in T$. Further, define $(f \vee g)(x) = f(x) \vee g(x)$ and let $e = 1$.

Example 3.3. Let X be the space of continuous real valued functions on an extremally disconnected compact Hausdorff space. Define « $\leqslant$ », « $\vee$ » and e as in Example 3.2 [**2**].

Example 3.4. Let $X = L^p(S, \mu)$ the space of equivalence classes of real valued p-summable functions where $p > 0$ and (S, μ) is a finite measure space. Define « $\leqslant$ » by $f \leqslant g$ if $f(x) \leqslant g(x)$ for almost all $x \in X$. Further, define « $\vee$ » and e as in Example 3.2 ([**1**], p. 106).

4 - Main results

Lemma 1. *Let X be an order complete o.l.s. Let Y be an o.l.s. and $W = \{w \in Y \mid w \geqslant 0\}$. Let M be a subspace of Y such that for each $y \in Y$*

$$(y + M) \cap W \neq \emptyset \quad \textit{if and only if} \quad (y + M) \cap (-W) \neq \emptyset .$$

Then each linear monotone mapping f from M into X has a linear monotone extension F from Y into X.

Proof. Zorn's lemma guarantees the existence of a maximal linear monotone extension f_1 of f from M to a subspace M_1 of Y. Using the intersection property of the hypothesis, it is easy to show that for each $y \in Y$

$$(y + M_1) \cap W \neq \emptyset \quad \text{if and only if} \quad (y + M_1) \cap (-W) \neq \emptyset .$$

We claim that $M_1 = Y$ and hence f_1 is the required extension F.

If the claim is false, then there is a $y \in Y$ such that $y \notin M_1$. Let $A = \{v \mid v \in M_1 \text{ and } v \leq y\}$ and $B = \{z \mid z \in M_1 \text{ and } y \leqslant z\}$. We observe that $A \neq \emptyset$ if and only if $B \neq \emptyset$. For, $v \in A$, if and only if $v \leqslant y$, if and only if $y - v \in W$, if and only if there is a $z \in M_1$ such that $-y + z \in W$, if and only if $-y \geqslant -z$, if and only if $y \leqslant z$, if and only if $z \in B$.

We consider the two cases $A = \emptyset$ and $A \neq \emptyset$. If $A = \emptyset$, then $B = \emptyset$. Let $c \in X$ be fixed, R denote the set of reals and $M_2 = \{v + ty \mid v \in M_1 \text{ and } t \in R\}$. Define f_2 mapping M_2 into X by $f_2(v + ty) = f_1(v) + tc$. f_2 is clearly a proper linear extension of f_1 and is monotone. For suppose $v_1 + t_1 y \leqslant v_2 + t_2 y$, where $v_1, v_2 \in M_1$ and $t_1, t_2 \in R$. If $t_1 \neq t_2$, then either

$$\frac{1}{t_2 - t_1}(v_1 - v_2) \leqslant y \qquad \text{or} \qquad \frac{1}{t_2 - t_1}(v_1 - v_2) \geqslant y .$$

Thus, $A \neq \emptyset$ or $B \neq \emptyset$ which is contrary to $A = \emptyset$. So $t_1 = t_2$, which implies $v_1 \leqslant v_2$. Hence, $f_1(v_1) \leqslant f_1(v_2)$ and

$$f(v_1) + t_1 c \leqslant f(v_2) + t_2 c .$$

Thus, $f_2(v_1 + t_1 y) = f_2(v_2 + t_2 y)$, i.e., f_2 is monotone. This contradicts the maximality of f_1.

If $A \neq \emptyset$, then $B \neq \emptyset$. Clearly $z \geqslant y \geqslant v$ for all $v \in A$ and $z \in B$. Thus, $f_1(z) \geqslant f_1(v)$ for all $v \in A$ and $z \in B$. Since X is order complete, $a = \sup\{f_1(v) \mid v \in A\}$ and $b = \inf\{f_1(z) \mid z \in B\}$ exist. Further, $a \leqslant b$ and we take c to be any element of satisfyng $a \leqslant c \leqslant b$. Let M_2 and f_2 be defined as above. Again it is clear that f_2 is a proper linear extension of f_1 from M_1 to M_2. Further, f_2 is monotone. For suppose $v_1 + t_1 y \leqslant v_2 + t_2 y$ where $v_1, v_2 \in M_1$ and $t_1, t_2 \in R$. If $t_1 = t_2$, the monotonicity of f_2 follows as above. If $t_1 \neq t_2$, assume without loss of generality that $t_1 < t_2$. Then

$$\frac{1}{t_2 - t_1}(v_1 - v_2) \leqslant y , \qquad \text{hence} \qquad \frac{1}{t_2 - t_1}(v_1 - v_2) \in A.$$

Thus $$f_1\left(\frac{1}{t_2 - t_1}(v_1 - v_2)\right) \leqslant c \qquad \text{and} \qquad f_1(v_1) - f_1(v_2) \leqslant (t_2 - t_1)c .$$

Therefore $$f_1(v_1) + t_1 c \leqslant f_1(v_2) + t_2 c \qquad \text{and} \qquad f_2(v_1 + t_1 y) \leqslant f_2(v_2 + t_2 y) .$$

Thus the monotonicity of f_2 is established and the maximality of f_1 is contradicted.

Theorem 1. *Let X be an order complete o.l.s. with an Archimedean unit e. Then there is a norm on X relative to which X has the Hahn-Banach norm extension property.*

Proof. First, we utilize Lemma 1 to establish the fact that X has the Hahn-Banach sublinear extension property. I.e., given a linear space Y, a subspace M of Y, a sublinear function p mapping Y into X and a linear mapping f of M into X such that $f(y) \leqslant p(y)$ for each $y \in M$, we show there is a linear extension F of f mapping Y into X such that $F(y) \leqslant p(y)$ for each $y \in Y$. Let

$$L = Y \times X, \quad E = \{(y, f(y)) \mid y \in M\}, \quad K = \{(y, x) \mid y \in Y,\ x \in X \text{ and } p(y) \leqslant x\}.$$

Then $W = K - E$ is a wedge in the linear space L relative to which L is an ordered linear space. Moreover, W/E is a wedge in the factor space L/E relative to which L/E is an ordered linear space.

Let $X^{\#} = \{(0, x) \mid x \in X$ and 0 is the additive identity of $Y\}$. Then for $x_1, x_2 \in X$, $(0, x_1) + E = (0, x_2) + E$ if and only if $(0, x_1 - x_2) \in E$, if and only if $f(0) = x_1 - x_2$, if and only if $x_1 - x_2 = 0$ and $x_1 = x_2$.

Hence the mapping

$$g\colon X^{\#}/E \to X \quad \text{given by} \quad g((0, x) + E) = x,$$

is well defined. Further, it is easily seen to be linear and monotonic.

Now we claim that for each member $(y, x) + E$ of L/E, the subspace $X^{\#}/E$ satisfies

$$\big(((y, x)+E)+X^{\#}/E\big) \cap W/E \neq \emptyset \text{ if and only if } \big(((y, x)+E)+X^{\#}/E\big) \cap (-W/E) \neq \emptyset.$$

For suppose there is an $x_1 \in X$ such that

$$(y, x) + E + (0, x\ ) + E \in W/E.$$

Since X has an Archimedean unit, there is an $x_2 \in X$ such that $x_2 > p(-y) + x$ i.e., $x_2 - x > p(-y)$. This means $(-y, x_2 - x) \in W$. But $(-y, x_2 - x) = (-y, -x) + (0, x_2)$. Thus $\big((-y, -x) + E\big) + \big((0, x_2) + E\big) \in W/E$ so that $\big((y, x) + E\big) + \big((0, -x_2) + E\big) \in (-W/E)$.

The implication in the opposite direction is established in a similar fashion. Hence, we can conclude from Lemma 1 that g has a linear monotone extension G from L/E into X.

Now let j be the mapping from L into L/E defined by

$$j((y, x)) = (y, x) + E .$$

Let H denote the composite mapping $G \circ j$. It is easy to show that H is a linear monotone mapping from L into X.

We note that for each $y \in Y$ there exists an $x \in X$ with $H((y, x)) = 0$. For let x_1 be an arbitrary member of X. Take $x = x_1 - H((y, x_1))$. Then

$$\begin{aligned} H((y, x)) &= H((y, x_1 - H(y, x_1))) = H((y, x_1) - (0, H(y, x_1))) \\ &= H((y, x_1)) - H((0, H(y, x_1))) = H((y, x_1)) - G((0, H(y, x_1)) + E) \\ &= H((y, x_1)) - H((y, x_1)) = 0 . \end{aligned}$$

Further, we note that for $y \in Y$ and $x_1, x_2 \in X$

$$H((y, x_1)) = 0 = H((y, x_2))$$

implies

$$H((y, x_1)) - H((y, x_2)) = H((0, x_1 - x_2)) = 0 .$$

But
$$H((0, x_1 - x_2)) = G((0, (x_1 - x_2) + E)) = x_1 - x_2 .$$

Hence, $x_1 = x_2$. Thus, we have established for each $y \in Y$ the existence of a unique $x \in X$ such that $H((y, x)) = 0$.

Define the mapping $F\colon Y \to X$ by $F(y) = x$, where $x \in X$ satisfies $H((y, x)) = 0$. For each pair $(y, f(y))$, $y \in M$, the pair $(y, f(y)) \in E$. Hence, $H((y, f(y))) = 0$ since E is in the kernel of H. Thus F extends f and clearly F is linear. Moreover, for each $y \in Y$, $F(y) \leqslant p(y)$. This is true for $y \in M$ by hypothesis. For $y \in X - M$, suppose $F(y) > p(y)$. Then

$$0 = H((y, F(y))) = H((y, p(y)) + (0, F(y) - p(y))) \geqslant H((0, F(y) - p(y)))$$

(since $(y, p(y))$ is in the wedge W of L and H is monotone). But

$$H((0, F(y) - p(y))) = F(y) - p(y) > 0 .$$

Hence, we have a contradiction.

Now we proceed to establish a norm on X relative to which it has the Hahn-Banach norm extension property.

The order on X gives rise to the order topology $\mathfrak{T}$, i.e., the finest locally, convex topology on X for which every order interval is bounded. It can be shown ([4], p. 251) that $(X, \mathfrak{T})$ is normable with a norm given by the gauge of e, i.e.,

$$\|x\| = p_e(x) = \inf\{\lambda > 0 \mid x \in [-\lambda e, \lambda e]\} .$$

Let Y be any linear space, M any subspace of Y and T any bounded linear operator mapping M into X. Consider the sublinear function $p(y) = \|T\|\,\|y\|e$. By definition,

$$\|T(y)\| = \inf\{\lambda > 0 \mid T(y) \in [-\lambda e, \lambda e]\} = \lambda_0 \leqslant \|T\|\,\|y\| .$$

Also, $[-e, e]$ is radial so that

$$T(y) \in [-\|T\|\,\|y\|e, \|T\|\,\|y\|e] , \qquad \text{i.e.,} \qquad T(y) \leqslant \|T\|\,\|y\|e = p(y) ,$$

for every $y \in M$.

Since X has the Hahn-Banach sublinear extension property, there is a T': $Y \to X$ with $T'(y) = T(y)$ for every $y \in M$ and $T'(y) \leqslant p(y)$ for every $y \in Y$. To show $\|T'\| = \|T\|$, it suffices to show $\|T'\| \leqslant \|T\|$. Now

$$T'(y) \leqslant \|T\|\,\|y\|e \qquad \text{and} \qquad -T'(y) = T'(-y) \leqslant \|T\|\,\|y\|e$$

so that

$$-\|T\|\,\|y\|e \leqslant T'(y) \leqslant \|T\|\,\|y\|e .$$

But by definition

$$\|T'(y)\| = \inf\{\lambda > 0 \mid T'(y) \in [-\lambda e, \lambda e]\} .$$

Hence,

$$\|T'(y)\| \leqslant \|T\|\,\|y\| \qquad \text{and} \qquad \|T'\| = \sup_{\|y\|=1} \|T'(y)\| \leqslant \|T\| ,$$

i.e., X has the Hahn-Banach norm extension property.

Corollary 1. *Each Banach space X with the Hahn-Banach norm extension property has a natural ordering with an Archimedean unit relative to which it has the Hahn-Banach sublinear extension property, and conversely, each order complete ordered linear space with an Archimedean unit which has the Hahn-Banach sublinear extension property, has a «natural» norm relative to which it has the Hahn-Banach norm extension property.*

P r o o f. The first statement in the corollary is an immediate consequence of Kelley's result [**3**] and the first part of the proof of Theorem 1. The converse statement follows from the second part of the proof of Theorem 1.

C o r o l l a r y 2. *Let X be a Banach space with dimension d (not necessarily finite). Let the unit ball B in X^*, the dual space of X, be topologized with the weak star topology. If each subspace of $C(B)$ with dimension d has a nonempty intersection with the cone of positive functions in $C(B)$, then X has the Hahn-Banach norm extension property.*

P r o o f. By the Banach-Alaoglu Theorem, B is a compact Hausdorff space. Furthermore, X is linearly isometric to a subspace of $C(B)$ ([**1**], p. 93). We identify X with this subspace. This gives X a natural ordering, viz., that of $C(B)$. Moreover, X is order complete relative to this ordering since X is a Banach space.

By hypothesis X contains a positive function e and since B is compact Hausdorff, e attains its minimum on B. Hence, there is a positive number a such that $e(x) \geqslant a$ for each $x \in B$. Clearly e is an Archimedean unit in X so by Theorem 1, there is some norm on X relative to which X has the Hahn-Banach norm extension property.

However, examining the construction of this norm in the proof of Theorem 1, we see that it is actually the sup norm which is the original norm on X under the identification.

References

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A b s t r a c t

In this paper it is shown that every order complete ordered linear space with an Archimedean unit has the Hahn-Banach norm extension property. A relation between the Hahn-Banach norm extension and sublinear extension properties follows from this result. Also, a condition under which a Banach space has the norm extension property is given.

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