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On the structure of overfilling sequences of a Banach space (**)

1 - Introduction

In this note theorems are enumerated by Roman figures, lemmas by Arabic figures and theorems of recalls by starred Roman figures. $\{n\}$ is the sequence of the natural numbers, R^+ the positive real semiaxis $(R^+ = \{t \in R;\ t > 0\})$, $\mathscr{C}$ the complex field, B a Banach space, B' the dual of B and S_B the *unit sphere* of B $(S_B = \{x \in B;\ \|x\| = 1\})$; moreover, if $\{x_n\} \subset B$, span $\{x_n\}$ is the linear manifold spanned by $\{x_n\}$ and $[x_n]$ is the closure of span $\{x_n\}$. We recall in **2** all the standard definitions, while in **3** we report the proofs of all the properties that we state in this paragraph.

We shall now only recall that a sequence $\{y_n\}$ of B is

(D_1) ([7], p. 113) *overfilling* if $[y_n] = [y_{n_k}]$, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$;

(D_2) [9]$_3$ $\overline{Y}$-*overfilling* if $\bigcap_{m=1}^{\infty} [y_{n_k}]_{k \geqslant m} = \overline{Y}$, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$;

(D_3) [9]$_2$ *convergent of order* p to $\{\bar{y}_k\}_{k=1}^{p} \subset S_B$, where $1 \leqslant p \leqslant +\infty$, if $\forall m$, with $1 \leqslant m \leqslant p$, $\exists \{v_{mn}\} \subset [\bar{y}_k]_{k=1}^{m-1}$ so that $\lim_{n\to\infty} (y_n + v_{mn})/\|y_n + v_{mn}\| = \bar{y}_m$;

(D_4) [9]$_1$ with *property* P if, $\forall y \in [y_n]$, $\exists$ two infinite complementary subsequences $\{y_{n_k}\}$ and $\{y_{n'_k}\}$ of $\{y_n\}$, which depend on y, so that $y \in [y_{n_k}] + [y_{n'_k}]$.

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About (D_1) we recall that any separable B has an overfilling sequence complete in B ([**3**] p. 193, [**7**] p. 113 (method of Ju. I. Lyubich) and $[\mathbf{9}]_2$). About (D_2) we remark that *$\{y_n\}$ is overfilling $\Leftrightarrow$ $\{y_n\}$ is $\overline{Y}$-overfilling with $\overline{Y} = [y_n]$*. About (D_3) we point out that, as well as for $[\mathbf{9}]_2$, also the overfilling sequences of [**3**] (p. 193) and of [**7**] (p. 113) have subsequences convergent of infinite order. Therefore examples of overfilling sequences, by the infinitely convergent sequences, are well known; instead the structure of the general overfilling sequence was still unknown.

Then we fill this gap, precisely we shall prove that *the overfilling sequences are just union of infinitely convergent sequences $\{y_n\}$, such that the sequence $\{\bar{y}_k\}$ of the limit points is complete in $[y_n]$*.

About (D_4) we remark that a minimal sequence has not in general property P $[\mathbf{9}]_1$, while an uniformly minimal sequence always has this property ($[\mathbf{9}]_4$, theorem VI), hence the minimal sequences with property P are intermediate between the minimal and the uniformly minimal sequences.

In what follows we shall be concerned with sequences that are union of an infinite number of sequences, then it is usefull the following definition

(D_5) $\{y_n\}$ is an *I-sequence* if $\exists$ an order relation $<$ among its elements so that, $\forall n$ and m with $n \neq m$, it is $y_n < y_m$ or $y_m < y_n$.

Let $\{y_n\}$ be an I-sequence, by (D_5) it follows that

(1) $\forall m$ $\exists$ two complementary subsequences $\{y_{ms_n}\}$ and $\{y_{md_n}\}$ of $\{y_n\}_{n\neq m}$, so that $y_{ms_n} < y_m < y_{md_n}$, $\forall n$.

In what follows $\{y_{ms_n}\}$ and $\{y_{md_n}\}$ will always be the sequences of (1). Our aim is now to single out the elementary sequences that form the structure of the sequences without an infinite basic subsequence.

Firstly we state a definition that regards a simple type of overfilling sequence.

(D_6) $\{y_n\}$ is *elementary of first type if*

(a)′ $\{y_n\}$ is overfilling;

(b)′ $\exists$ an I-sequence $\{\bar{y}_k\}$ of S_B, complete in $[y_n]$ and with $\bar{y}_m \notin [\bar{y}_{ms_n}]$ $\forall m$, so that $\{y_n\}$ is infinitely convergent to $\{\bar{y}_k\}$ (that is, $\forall m$, $\exists \{v_{mn}\} \subset [\bar{y}_{ms_n}]$ so that $\lim\limits_{n\to\infty} (y_n + v_{mn})/\|y_n + v_{mn}\| = \bar{y}_m$).

More complicated is the investigation of the simple types of minimal sequences without an infinite basic subsequence, because we have to distinguish three subcases:

(D_7) $\{y_n\}$ is *elementary of second type* if

(a)″ $\{y_n\}$ is minimal and $\overline{Y}$-overfilling, with dimension of $\overline{Y} = p \geqslant 1$;

(b)″ we have one of the following three mutually exclusive alternatives;

(b_1) $\exists$ an I-sequence $\{\bar{y}_k\}_{k=1}^p$ of S_B, complete in $\bar{Y}$ and with $\bar{y}_m \notin [\bar{y}_{ms_n}]$ $\forall m$, so that $\{y_n\}$ is convergent of order p to $\{\bar{y}_k\}_{k=1}^p$;

(b_2) (possible only for $p = 1$) $\exists \bar{y} \neq 0$, $\{\bar{\alpha}_n\} \subset \mathscr{C}$ with $\lim_{n\to\infty} \bar{\alpha}_n = 1$, a basic sequence $\{y_n^*\}$ of B with $\inf_n \|y_n^*\| > 0$ and weakly convergent to 0, so that $y_n/\|y_n\| = \bar{\alpha}_n \bar{y} + y_n^*$, $\forall n$;

(b_3) (it is a combination of (b_1) and (b_2)) $\exists$ an I-sequence $\{\bar{y}_k\}$ of S_B and $\bar{y} \neq 0$, with $\{\bar{y}_k\} \cup \bar{y}$ complete in $\bar{Y}$, $\bar{y}_m \notin [\bar{y}_{ms_n}]$, $\forall m$ and $\bar{y} \notin [\bar{y}_k]$, so that $\{y_n\}$ is convergent of order $p-1$ to $\{\bar{y}_k\}$; moreover, in the Banach space $B/[\bar{y}_k]$, $\exists \{\bar{\alpha}_n\} \subset \mathscr{C}$ with $\lim_{n\to\infty} \bar{\alpha}_n = 1$, a basic sequence $\{y_n^*\}$ of B, with $\inf_n \|y_n^* + [\bar{y}_k]\| > 0$, $\{y_n^* + [\bar{y}_k]\}$ basic and weakly convergent to 0, so that

$$(y_n + [\bar{y}_k])/\|y_n + [\bar{y}_k]\| = (\bar{\alpha}_n \bar{y} + y_n^*) + [\bar{y}_k], \forall n.$$

The main result of this note is the following theorem (where we leave out the trivial case of $[y_n]$ finite dimensional subspace of B).

I. *Every sequence $\{y_n\}$ of B has an infinite subsequence $\{y_{n_k}\}$ satisfying one of the following three mutually exclusive alternatives*:

(a) *$\{y_{n_k}\}$ is elementary of first type*;

(b) *$\{y_{n_k}\}$ has property P and is elementary of second type*;

(c) *$\{y_{n_k}\}$ is basic.*

It is obvious that the overfilling sequences and the basic sequences have always property P, hence the following immediate corollary of theorem I answers a question raised by author ([**9**]$_4$, problems 1 and 2).

Corollary I. *Every infinite sequence of B has an infinite subsequence with property P.*

We pass now to consider the stability of preceding properties for sufficiently « near » sequences. We call *limit I-sequence* the I-sequence $\{\bar{y}_k\}$ of the limit points of (D_6) and (D_7), then we state that

II. *Let $\{y_n\}$ be an elementary sequence of B, with $\{\bar{y}_k\}$ limit I-sequence: $\Rightarrow \exists \{\varepsilon_n\} \subset R^+$ so that, $\forall \{x_n\} \subset B$ with $\|x_n - y_n\| < \varepsilon_n$, $\forall n$, $\{x_n\}$ has an elementary subsequence with $\{\bar{y}_k\} \cup \{\bar{x}_k\}$ limit I-sequence, where $\bar{y}_k < \bar{x}_n$, $\forall k$ and n.*

A minimal sequence with property P is stable for « near » sequences ([**9**]$_4$ theorem III); therefore, in theorem I, the sequences of (b) and (c) are stable for « near » sequences. Instead this is not true for (a) of theorem I, because every $\{y_n\}$, with $[y_n]$ infinite codimensional subspace of B, can be approximated as we want by minimal sequences of B ([**9**]$_4$ theorem IV). Then by theorem II it immediately follows that

Corollary II. *Let* $\{y_n\}$ *be an elementary sequence of* B, *then* $\exists\{\varepsilon_n\}\subset R^+$ *so that,* $\forall\{x_n\}\subset B$ *with* $\|x_n - y_n\| < \varepsilon_n$ $\forall n$, *we have the following implications:*

(a) $\{y_n\}$ *is of 2-nd type (and has property* P) $\Rightarrow$ $\{x_n\}$ *has an elementary subsequence of 2-nd type (and has property* P);

(b) $\{y_n\}$ *is of 1-st type and* $\{x_n\}\subset[y_n]$ $\Rightarrow$ $\{x_n\}$ *is elementary of 1-st type and complete in* $[y_n]$.

Finally let us raise a few questions.

It is well known ([5], see also [6] p. 116) that, if $\{x_n\}\subset B$ with $x_m\notin \text{span }\{x_n\}_{n=1}^{m-1}$ $\forall m>1$, then $\exists$ an M-basis $\{y_n\}$ of $[x_n]$ with $\text{span }\{x_n\}_{n=1}^{m} = \text{span }\{y_n\}_{n=1}^{m}$ $\forall m$. We raise the question if it is possible to extend this property to the I-sequences, that is

Problem 1. Let $\{x_n\}$ be an I-sequence of B, with $x_m\notin[x_{ms_n}]$, $\forall m$, does it exist an M-basic I-sequence $\{y_n\}$ so that $[y_{ms_n}]=[x_{ms_n}]$ $\forall m$?.

It is possible to verify that this problem is equivalent to next problem, which concerns an improvement of theorem I.

Problem 2. Is it possible to get M-basic the limit I-sequence of (a) and (b) of theorem I?

2 - Standard definitions and recalls

Let $\{x_n\}\subset B$ and $\{f_n\}\subset B'$, we recall that

(D_8) $\{x_n\}$ is *minimal* if $x_m\notin[x_n]_{n\neq m}$, $\forall m$;

(D_9) $\{x_n\}$ is *uniformly minimal* if $\inf_m \{\inf\{\|x_m + x\|;\ x\in\text{span }\{x_n\}_{n\neq m}\}\}>0$;

(D_{10}) $\{x_n, f_n\}$ is a *biorthogonal system* if $f_m(x_n)=\delta_{mn}$ $\forall m$ and n.

Let $\{x_n, f_n\}$ be a biorthogonal system of B, we recall that

(D_{11}) $\{x_n\}$ is *M-basis* of B if $[f_n]$ is *total on* $[x_n]$ (that is $[x_n]\cap[f_n]_\perp=\{0\}$, where $[f_n]_\perp=\{x\in B;\ f_n(x)=0\ \forall n\}$) and if $[x_n]=B$;

(D_{12}) $\{x_n\}$ is *basis* of B if $x=\sum_1^\infty{}_n f_n(x)x_n$, $\forall x\in B$;

(D_{13}) $\{x_n\}$ is *M-basic* (*basic*) *sequence* if $\{x_n\}$ is M-basis (basis) of $[x_n]$;

(D_{14}) $\{x_n, f_n\}$ is *bounded* if $\|x_n\|\cdot\|f_n\|\leqslant M<+\infty$ $\forall n$.

Moreover we recall that

(a) ([**5**], see also [**8**]$_2$ p. 54) $\{x_n\}$ *minimal* $\Leftrightarrow \exists\{f_n\} \subset B'$ *with* $\{x_n, f_n\}$ *biorthogonal system*;

(b) ([**1**], see also [**8**]$_1$ p. 165) $\{x_n\}$ *uniformly minimal* $\Leftrightarrow \{f_n\} \subset B'$ *so that* $\{x_n, f_n|_{[x_k]}\}$ *is a bounded biorthogonal system of* $[x_k]$.

3 - Lemmas and proofs of theorems

Let $\{y_n\}$ be an infinite sequence of B, then

(D_{15}) we call *nucleus* of $\{y_n\}$ the set $N\{y_n\} = \{y \in [y_n]$ so that, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$, it is $y \in [y_{n_k}]\}$;

(D_{16}) we say that $\{y_n\}$ is *denucleated* if $N\{y_{n_k}\} = 0$, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$.

1 – *Let* $\{x_n\}$ *be an infinite sequence of* B, *then*: $\Rightarrow \exists$ *an infinite subsequence* $\{y_n\}$ *of* $\{x_n\}$, *so that* $N\{y_n\} = N\{y_{n_k}\}$, $\forall$ *infinite subsequence* $\{y_{n_k}\}$ *of* $\{y_n\}$.

Proof. Let U be the set of all the infinite subsequences of $\{x_n\}$. If $u_1 = \{x_{1,n}\}$ and $u_2 = \{x_{2,n}\}$ are two elements of U, let us set in U the following order relation

$$(2) \qquad \begin{aligned} &u_1 \leqslant u_2 \quad \text{if } \exists\{\bar n_1, \bar n_2\} \subset \{n\} \text{ so that } \{x_{2,n}\}_{n \geqslant \bar n_2} \subseteq \{x_{1,n}\}_{n \geqslant \bar n_1}; \\ &u_1 < u_2 \ \text{ if } u_1 \leqslant u_2, \text{ moreover } N\{x_{1,n}\} \subset N\{x_{2,n}\}. \end{aligned}$$

Let V be a totally ordered subset of U, then let us set

$$(3) \qquad N = \{x \in N\{x_{1,n}\};\ \{x_{1,n}\} \in V\}.$$

Let now $\{z_n\} \subset B$ so that

$$(4) \qquad \{z_n\} \subset N, \quad \text{moreover} \quad N \subseteq [z_n].$$

By (3) and (4) we have that

$$(5) \qquad \exists\{u_n\} \subset V, \text{ with } u_n = \{\tilde x_{n,k}\}_{k=1}^{\infty}\ \forall n, \text{ so that } z_n \in N\{\tilde x_{n,k}\}_{k=1}^{\infty},\ \forall n.$$

Now V is totally ordered, hence let us set

$$v_n = \max\{u_k;\ 1 \leqslant k \leqslant n\}, \qquad \forall n. \tag{6}$$

By (5) and (6) it follows that

$$\{z_i\}_{i=1}^n \subset N\{x_{n,k}\}_{k=1}^\infty, \text{ where } \{x_{n,k}\}_{k=1}^\infty = v_n \in V, \text{ with } v_n \leqslant v_{n+1}, \ \forall n. \tag{7}$$

We have now two possibilities

(a) Suppose that $\exists \tilde{v} \in U$ so that

$$\tilde{v} = \{x_{\tilde{n}_k}\} \in V, \qquad \text{with } v_n \leqslant \tilde{v}, \ \forall n. \tag{8}$$

By (2), (7) and (8) $\{z_n\} \subset N\{x_{\tilde{n}_k}\}$, on the other hand the nucleus of a sequence is a subspace of B, hence $N \subseteq [z_n] \subseteq N\{x_{\tilde{n}_k}\}$ by (4); but, by (3), $N\{x_{\tilde{n}_k}\} \subseteq N$, because $\tilde{v} \in V$ by (8), then

$$N = N\{x_{\tilde{n}_k}\}. \tag{9}$$

Let now $v \in V$, it is $v \leqslant \tilde{v}$ or $\tilde{v} < v$ because V is totally ordered; but, by (2), (3), (8) and (9), it is impossible that $\tilde{v} < v$; hence $v \leqslant \tilde{v}$, that is $\tilde{v}$ is a maximal element of V.

(b) Suppose now that the element $\tilde{v}$ of (8) does not exist, then we have that

$$\forall v \in V, \qquad \exists m \in \{n\} \text{ so that } v \leqslant v_m. \tag{10}$$

Let us set $w = \{x_{n,n}\}_{n=1}^\infty$, that is $\{x_{n,n}\}$ is the diagonal subsequence of the sequences $\{x_{n,k}\}_{k=1}^\infty$ of (7); by (2) and (7) we have that

$$v_n \leqslant w \qquad \forall n. \tag{11}$$

Therefore by (10) and (11) it follows that $v \leqslant w$, $\forall v \in V$, that is w is a majorant element of V.

Consequently, by lemma of Zorn, by (a) and (b) it follows that U has a maximal element, that is $\exists$ an infinite subsequence $\{y_n\} = \bar{v}$ of $\{x_n\}$, so that it is impossible that $\bar{v} < u$ for $u \in U$, that is, by (2), $N\{y_{n_k}\} = N\{y_n\}$, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$. This completes the proof of Lemma 1.

2 – *Let $\{x_n\}$ be an infinite sequence of B, without an infinite basic subsequence, then:* $\Rightarrow \exists$ *an $\overline{Y}$-overfilling subsequence $\{y_n\}$ of $\{x_n\}$, with dimension of $\overline{Y} \geqslant 1$.*

Proof. Firstly let us recall that ([2], see also [7] p. 128 and [$\mathbf{9}_4$] Table 2)

(12) $\{x_n\}$ denucleated $\Leftrightarrow$ every infinite subsequence of $\{x_n\}$ has an infinite basic subsequence.

Then, by (12) and (D_{16}), $\exists$ an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with dimension of $N\{x_{n_k}\} \geqslant 1$; consequently, by Lemma 1, $\exists$ an infinite subsequence $\{y_n\}$ of $\{x_{n_k}\}$ so that

(13) $N\{y_{n_k}\} = \overline{Y}$, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$, with dimension of $\overline{Y} \geqslant 1$.

Let us consider the sequence $\{y_n + \overline{Y}\}$ of $B/\overline{Y}$.

Suppose that $\exists$ an infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that

$$\bar{x} + \overline{Y} \in N\{y_{n_k} + \overline{Y}\}, \qquad \text{with} \quad \bar{x} + \overline{Y} \neq 0\,. \tag{14}$$

Let $\{y_{n'_k}\}$ be an infinite subsequence of $\{y_{nk}\}$, then $\bar{x} + \overline{Y} \in [y_{n'_k} + \overline{Y}]$ by (14) and (D_{15}), but $\overline{Y} \subset [y_{n'_k}]$ by (13), hence it is easy to see that

$$\bar{x} \in [y_{n'_k}]\,. \tag{15}$$

Therefore by (15) $\bar{x} \in N\{y_{n_k}\}$, but this is absurd because, by (13) and (14), $\bar{x} + \overline{Y} \neq 0 \Rightarrow \bar{x} \notin N\{y_{n_k}\}$; hence (14) is not possible, that is $\{y_n + \overline{Y}\}$ is denucleated. Consequently, by (12), $\exists$ an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, so that $\{y_n + \overline{Y}\}$ is basic, hence

$$\bigcap_{m=1}^{\infty} [y_n + \overline{Y}]_{n>m} = \{0\}\,. \tag{16}$$

Suppose now that $\bar{y} \in \bigcap_{m=1}^{\infty} [y_n]_{n>m}$, then $\bar{y} + \overline{Y} \in \bigcap_{m=1}^{\infty} [y_n + \overline{Y}]_{n>m}$, consequently, by (16), $\bar{y} + \overline{Y} = 0$, that is $\bar{y} \in \overline{Y}$, therefore

$$\bigcap_{m=1}^{\infty} [y_n]_{n>m} \subseteq \overline{Y}\,. \tag{17}$$

On the other hand, $\forall$ infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$, by (17) it is

$$\bigcap_{m=1}^{\infty} [y_{n_k}]_{k>m} \subseteq \bigcap_{m=1}^{\infty} [y_n]_{n>m} \subseteq \overline{Y}\,.$$

But $\overline{Y} \subseteq \bigcap_{m=1}^{\infty} [y_{n_k}]_{k>m}$ by (13), therefore

$$\bigcap_{m=1}^{\infty} [y_{n_k}]_{k>m} = \overline{Y}, \quad \forall \text{ infinite subsequence } \{y_{n_k}\} \text{ of } \{y_n\}.$$

This, by (D_2) and (13), completes the proof of Lemma 2.

Let us now recall a few theorems.

I* ($[\mathbf{9}]_2$, theorem V). *Let $\{y_n\}$ be an $\overline{Y}$-overfilling sequence of B, with dimension of $\overline{Y} = p > 1$, then: $\Rightarrow \exists$ an infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and an M-basic sequence $\{\bar{y}_k\}_{k=1}^{q}$ of S_B, with $p-1 \leqslant q \leqslant p$, so that $\{y_{n_k}\}$ is convergent of order q to $\{\bar{y}_k\}_{k=1}^{q}$.*

II* ($[\mathbf{9}]_3$, theorem IX). *Let $\{y_n\}$ be an $\overline{Y}$-overfilling sequence of B (also convergent of infinite order) then: $\{y_n\}$ has an infinite minimal subsequence $\Leftrightarrow \overline{Y}$ is an infinite codimensional subspace of $[y_n]$.*

III* ($[\mathbf{9}]_2$, lemma 1). *Let $\{y_n\}$ be an infinite sequence of S_B, weakly and not strongly convergent to $\bar{y}$, then: $\Rightarrow \exists$ an infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$, $\{\bar{\alpha}_n\} \subset \mathscr{C}$ with $\lim_{n\to\infty} \bar{\alpha}_n = 1$, a basic sequence $\{y_n^*\}$ with $\inf_n \|y_n^*\| > 0$ and weakly convergent to 0, so that $y_{n_k} = \bar{\alpha}_k \bar{y} + y_k^*$, $\forall k$.*

3 – *Let $\{x_n\}$ be an $\overline{Y}$-overfilling sequence of B, with dimension of $\overline{Y} > 1$, then: $\Rightarrow \exists$ an infinite subsequence $\{y_n\}$ of $\{x_n\}$, an I-sequence $\{\bar{y}_k\}_{k=1}^{q}$ of S_B with $\bar{y}_m \notin [\bar{y}_{ms_n}]$ $\forall m$ and with codimension of $[\bar{y}_k]_{k=1}^{q}$ in $\overline{Y} \leqslant 1$, so that $\{y_n\}$ is convergent of order q to $\{\bar{y}_k\}$.*

Proof. If $\overline{Y}$ has finite dimension, the thesis immediately follows by theorem I*. Therefore suppose that $\overline{Y}$ has infinite dimension, by theorem I* $\exists\{\tilde{y}_{1,n}\} \subset S_B$ and an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$, so that

$$\forall m,\ \tilde{y}_{1,m} \notin \operatorname{span}\{y_{1,n}\}_{n=1}^{m-1}, \quad \text{moreover } \exists\{\tilde{v}_{mk}\} \subset \operatorname{span}\{\tilde{y}_{1,n}\}_{n=1}^{m-1} \tag{18}$$

so that $\lim_{k\to\infty} (x_{n_k} + \tilde{v}_{mk})/\|x_{n_k} + \tilde{v}_{mk}\| = \tilde{y}_{1,m}$.

Let us set $\overline{Y}_1 = [\tilde{y}_{1,n}]$; if the codimension of $\overline{Y}_1$ in $\overline{Y}$ is $\leqslant 1$ the thesis is proved, therefore suppose that the codimension is > 1.

By (18) it follows that $\exists\{\bar{y}_{1,n}\} \subset B$ so that

$$\begin{aligned} &\|\bar{y}_{1,m}\| = 1, \ \operatorname{span}\{\bar{y}_{1,n}\}_{n=1}^{m} = \operatorname{span}\{\tilde{y}_{1,n}\}_{n=1}^{m}, \\ &\inf\{\|\bar{y}_{1,m} + y\|;\ y \in \operatorname{span}\{\tilde{y}_{1,n}\}_{n=1}^{m-1}\} > 3/4,\ \forall m. \end{aligned} \tag{19}$$

By (19), $\forall m$, $\bar{y}_{1,m} = \bar{\alpha}_m \tilde{y}_{1,m} + \hat{y}_m$, with $\hat{y}_m \in \text{span}\,\{\tilde{y}_{1,n}\}_{n=1}^{m-1}$; consequently, setting $v_{mk} = \tilde{v}_{mk} + \hat{y}_m \|x_{n_k} + \tilde{v}_{mk}\|/\bar{\alpha}_m, \forall k$, by (18) it follows that (without losing of generality we can suppose $\bar{\alpha}_m \in R^+$)

$$\lim_{k\to\infty} \frac{x_{n_k} + v_{mk}}{\|x_{n_k} + v_{mk}\|} = \lim_{k\to\infty} \frac{\bar{\alpha}_m x_{n_k} + \bar{\alpha}_m \tilde{v}_{mk} + \hat{y}_m \|x_{n_k} + \tilde{v}_{mk}\|}{\|\bar{\alpha}_m x_{n_k} + \bar{\alpha}_m \tilde{v}_{mk} + \hat{y}_m \|x_{n_k} + \tilde{v}_{mk}\|\,\|}$$

$$= \lim_{k\to\infty} (\bar{\alpha}_m \frac{x_{n_k} + \tilde{v}_{mk}}{\|x_{n_k} + \tilde{v}_{mk}\|} + \hat{y}_m) / \|\bar{\alpha}_m \frac{x_{n_k} + \tilde{v}_{mk}}{\|x_{n_k} + \tilde{v}_{mk}\|} + \hat{y}_m\| = \frac{\bar{y}_{1,m}}{\|\bar{y}_{1,m}\|} = \bar{y}_{1,m}\,.$$

Therefore by (D_3) and (19) we have that

(20) $\{x_{n_k}\}$ is infinitely convergent to $\{\bar{y}_{1,n}\}$,

where $\inf\{\|\bar{y}_{1,m} + y\|;\ y \in \text{span}\,\{\bar{y}_{1,n}\}_{n=1}^{m-1}\} > 3/4$, $\forall m$.

Let us consider the sequence $\{x_{n_k} + \bar{Y}_1\}$ of $B/\bar{Y}_1$; we supposed the codimension of $\bar{Y}_1$ in $\bar{Y} > 1$, hence by hypothesis $\{x_{n_k} + \bar{Y}_1\}$ is $\bar{X}_1$-overfilling, with dimension of $\bar{X}_1 > 1$. Therefore, by theorem I*, $\exists\{\hat{y}_{2,k}\}_{k=1}^{q} \subset B$ and an infinite subsequence $\{x_{n'_k}\}$ of $\{x_{n_k}\}$ so that $p-1 \leqslant q \leqslant p \leqslant +\infty$, where p is the dimension of $\bar{X}_1$; moreover, $\forall m$ with $1 \leqslant m \leqslant q$, $\hat{y}_{2,m} + \bar{Y}_1 \notin \text{span}\,\{\hat{y}_{2,k} + \bar{Y}_1\}_{k=1}^{m-1}$ and $\exists\{v_{2mk} + \bar{Y}_1\} \subset \text{span}\,\{\hat{y}_{2,k} + \bar{Y}_1\}_{k=1}^{m-1}$ so that

$$\lim_{k\to\infty} \frac{(x_{n'_k} + v_{2mk}) + \bar{Y}_1}{\|(x_{n'_k} + v_{2mk}) + \bar{Y}_1\|} = \hat{y}_{2,m} + \bar{Y}_1\,. \tag{21}$$

By (D_3), (20) and (21), setting $\tilde{y}_{2,n} = \hat{y}_{2,n}/\|\hat{y}_{2,n}\|$ for $1 \leqslant n \leqslant q$, it follows that $\{x_{n'_k}\}$ is infinitely convergent to $\{\bar{y}_{1n}\}_{n=1}^{\infty} \cup \{\tilde{y}_{2,n}\}_{n=1}^{q}$, where

$$\tilde{y}_{2,m} \notin [\bar{y}_{1,n}] + \text{span}\,\{\tilde{y}_{2,n}\}_{n=1}^{m-1}\,, \qquad \forall m\,. \tag{22}$$

Proceeding as for (18), (19) and (20), by (22) it follows that

(23) $\{x_{n'_k}\}$ is infinitely convergent to $\{\bar{y}_{1,n}\}_{n=1}^{\infty} \cup \{\bar{y}_{2,n}\}_{n=1}^{q}$; moreover for $1 \leqslant m \leqslant q$, $[\bar{y}_{1,n}] + \text{span}\,\{\tilde{y}_{1,n}\}_{n=1}^{m} = [\bar{y}_{1,n}] + \text{span}\,\{\bar{y}_{1,n}\}_{n=1}^{m}$ and $\inf\{\|\bar{y}_{2,m} + y\|;$ $y \in [\bar{y}_{1,n}] + \text{span}\,\{\bar{y}_{2,n}\}_{n=1}^{m-1}\} > 3/4$.

Now, if p (hence q) of (21) is finite, or if $q = +\infty$ but $\bar{Y}_2 = [\{\bar{y}_{1,n}\} \cup \{\bar{y}_{2,n}\}]$ has codimension $\leqslant 1$ in $\bar{Y}$, the thesis is proved; otherwise, if we consider the sequence $\{x_{n'_k} + \overline{Y}_2\}$ of $B/\overline{Y}_2$, this sequence is $\overline{X}_2$-overfilling, with dimension of $\overline{X}_2 > 1$, hence we proceed as for $\{x_{n_k} + \overline{Y}_1\}$ and so on.

Let us call $\{\bar{y}_i,\ i \in I\}$ the set of all the limit points that we can find by (20), (23) and so on.

We remark that, by construction, if $i > j$[1], $\|\bar{y}_i + \alpha\bar{y}_j\| > 3/4$, $\forall\alpha \in \mathscr{C}$; moreover $\|\bar{y}_j + \alpha\bar{y}_i\| > 1/4$, $\forall\alpha\in\mathscr{C}$, otherwise, if $\|\bar{y}_j + \bar{\alpha}\bar{y}_i\| \leqslant 1/4$, it would be $|\bar{\alpha}| > 3/4$ (because $\|\bar{y}_i\| = \|\bar{y}_j\| = 1$), hence $\|\bar{y}_i + \bar{y}_j/\bar{\alpha}\| \leqslant 1/(4|\bar{\alpha}|) < 1/3$, absurd because $i > j$. Consequently

$$(24)\qquad \forall i \in I \text{ it is } \inf\{\|\bar{y}_i + \alpha\bar{y}_j\|,\ \alpha\in\mathscr{C} \text{ and } j \in I \text{ with } i \neq j\} > 1/4\,.$$

Now $\{\bar{y}_i,\ i\in I\} \subset [x_n]$, which is separable, hence $\exists$ a sequence $\{z_n\}$ dense in $[x_n]$; therefore, $\forall i \in I$, $\exists$ a natural number $n(i)$ so that

$$(25)\qquad \|\bar{y}_i + z_{n(i)}\| < 1/8\,, \qquad \forall i \in I\,.$$

On the other hand, by (24) and (25), it follows that

$$(26)\qquad \begin{gathered}\forall i, j \in I\,, \qquad \text{with } i \neq j\,,\\ \|\bar{y}_j + z_{n(i)}\| \geqslant \|\bar{y}_j - \bar{y}_i\| - \|\bar{y}_i + z_{n(i)}\| > 1/4 - 1/8 = 1/8\,.\end{gathered}$$

By (25) and (26) we have a correspondence one to one between $\{\bar{y}_i,\ i\in I\}$ and a subsequence $\{z_{n(i)}\}$ of $\{z_n\}$; that is $\{\bar{y}_i,\ i \in I\}$ is a sequence $\{\bar{y}_k\}$.

Let us now say that, in $\{\bar{y}_k\}$, $\bar{y}_{m_1} < \bar{y}_{m_2}$ if, in the construction of (20), (23) and so on, we meet the limit point $\bar{y}_{m_1}$ before of $\bar{y}_{m_2 i}$; therefore, by (D_5), (20), (23) and so on, we have that

(27) $\{\bar{y}_k\}$ is an I-sequence, with $\bar{y}_m \notin [\bar{y}_{ms_n}]$, $\forall m$; $\forall m$ $\exists$ a subsequence $\{x_{mn}\}$ of $\{x_n\}$ and $\{v_{mn}\} \subset [\bar{y}_{ms_n}]$ so that $\lim_{n\to\infty} (x_{mn} + v_{mn})/\|x_{mn} + v_{mn}\| = \bar{y}_m$; moreover $\bar{y}_{m_1} < \bar{y}_{m_2}$ implies that $\{x_{m_2 n}\}$ is a subsequence of $\{x_{m_1 n}\}$.

If the I-sequence $\{\bar{y}_k\}$ has not a last element let us set

(28) $x_{k(n),n} = y_n$, where $k(n)$ is the element of $\{i\}_{i=1}^n$, so that $\bar{y}_i < \bar{y}_{k(n)}$ for $1 \leqslant i(\neq k(n)) \leqslant n$, $\forall n$.

Let us fix m, by (27) and (28) $\{y_n\}_{n\geqslant m}$ is a subsequence of $\{x_{mn}\}_{n\geqslant m}$, therefore by (27) $\exists \{v'_{mn}\} \subseteq \{v_{mn}\}$ so that $\lim_{n\to\infty} (y_n + v'_{mn})/\|y_n + v'_{mn}\| = \bar{y}_m$. Instead, if the I-sequence $\{\bar{y}_k\}$ has a last element $\bar{y}_{\bar{k}}$, it is sufficient to set

$$(29) \qquad x_{\bar{k}n} = y_n \qquad \forall n .$$

By (28) and (29) we have that

(30) $\exists$ an infinite subsequence $\{y_n\}$ of $\{x_n\}$, which is infinitely convergent to the I-sequence $\{\bar{y}_k\}$.

Suppose now that the codimension of $[\bar{y}_k]$ in $\bar{Y}$ is > 1, by hypothesis and by (30) it would follow that the sequence $\{y_n + [\bar{y}_k]\}$ of $B/[\bar{y}_k]$ would be $\bar{V}$-overfilling, with dimension of $\bar{V} > 1$; therefore, by theorem I*, proceeding as for (21), (22) and (23), $\{\bar{y}_k\}$ would not be the I-sequence of all the limit points.

Then the codimension of $[\bar{y}_k]$ in $\bar{Y}$ is $\leqslant 1$, which completes the proof of Lemma 3.

4 – *Let $\{x_n\}$ be an $\bar{Y}$-overfilling sequence of B, with $[x_n]$ infinite dimensional subspace of B and with $\bar{Y}$ finite codimensional subspace of $[x_n]$, then: $\Rightarrow$ $\{x_n\}$ has an infinite subsequence $\{y_n\}$ which is elementary of 1^{st} type.*

Proof. By hypothesis $\exists$ a subspace V of $[x_n]$ so that

$$(31) \qquad V \cap \bar{Y} = \{0\} \text{ and } V + \bar{Y} = [x_n], \text{ with dimension of } V = p < +\infty .$$

By (31) we have that

$$(32) \qquad x_n = v_n + \tilde{y}_n , \quad \text{with} \quad v_n \in V \quad \text{and} \quad \tilde{y}_n \in \bar{Y} , \qquad \forall n .$$

Suppose that $\exists$ an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that

$$(33) \qquad v_{n_k} \text{ (of (32))} \neq 0 \qquad \forall k .$$

V has finite dimension, hence by (32) and (33) $\exists \bar{v} \in S_B$ and an infinite subsequence $\{n'_k\}$ of $\{n_k\}$ so that

$$(34) \qquad \lim_{k\to\infty} \frac{x_{n'_k} - \tilde{y}_{n'_k}}{\|x_{n'_k} - y_{n'_k}\|} = \lim_{k\to\infty} \frac{v_{n'_k}}{\|v_{n'_k}\|} = \bar{v} \in V .$$

By (34), $\forall$ infinite subsequence $\{x_{n''_k}\}$ of $\{x_{n'_k}\}$, $\bar{v} \in \bigcap_{m=1}^{\infty} [x_{n''_k}]_{k>m}$, but this is absurd by (D_2), because $\bar{v} \notin \bar{Y}$ by (31) and $\{x_{n''_k}\}$ is $\bar{Y}$-overfilling. Therefore (33) is not possible, that is $\exists n_0 \in \{n\}$ so that $v_n = 0$ for $n \geqslant n_0$, hence by (32)

$$\{x_n\}_{n \geqslant n_0} \subset \overline{Y} \, . \tag{35}$$

Now $\{x_n\}_{n \geqslant n_0}$ is $\overline{Y}$-overfilling, hence by (35), (D_1), (D_2) and by Lemma 3 we have that

(36) $\exists$ an infinite subsequence $\{y_n\}$ of $\{x_n\}_{n \geqslant n_0}$ which is overfilling and infinitely convergent to an I-sequence $\{\bar{y}_n\}$, with codimension of $[\bar{y}_n]$ in $\overline{Y}(= [y_n]) \leqslant 1$.

We remark that, by Lemma 3, $\{\bar{y}_n\}$ is the I-sequence of all the limit points of $\{y_n\}$. Suppose by absurd that $[\bar{y}_n]$ has codimension 1 in $\overline{Y}$, hence $\exists \bar{x}$ so that

$$\overline{Y} = [\bar{y}_n] + \operatorname{span}\{\bar{x}\}, \qquad \text{with } \bar{x} \notin [\bar{y}_n] \, . \tag{37}$$

By (36) and (37) we have that

$$y_n = \alpha_n \bar{x} + \hat{y}_n \, , \qquad \text{with } \hat{y}_n \in [\bar{y}_k] \, , \quad \forall n \, . \tag{38}$$

It is impossible that $\alpha_{n_k} = 0$ for an infinite subsequence $\{n_k\}$ of $\{n\}$, otherwise $[y_{n_k}] \subseteq [\bar{y}_n]$ (by (38)) $\subset \overline{Y}$ (by (37)) $= [y_{n_k}]$ (by (36)); therefore $\exists n_1 \in \{n\}$ so that

$$\alpha_n \neq 0 \qquad \text{for } n \geqslant n_1 \, . \tag{39}$$

By (38) and (39) it follows that $\exists$ an infinite subsequence $\{y_{n'_k}\}$ of $\{y_n\}$ so that

$$\lim_{k \to \infty} \frac{y_{n'_k} - \hat{y}_{n'_k}}{\|y_{n'_k} - \hat{y}_{n'_k}\|} = \lim_{k \to \infty} \frac{\alpha_{n'_k} \bar{x}}{\|\alpha_{n'_k} \bar{x}\|} = \bar{y} \in \operatorname{span}\{\bar{x}\} \, . \tag{40}$$

By (36), (38) and (40) it follows that $\bar{y}$ is a limit point of $\{y_{n'_k}\}$, absurd because by (37) and (40) $\bar{y} \notin [\bar{y}_n]$, while by Lemma 3 $\{\bar{y}_n\}$ is the I-sequence of all the limit points. Therefore (37) is not possible, that is $[\bar{y}_n] = \overline{Y} = [y_n]$, which completes the proof of Lemma 4.

Proof of Theorem 1. If $\{y_n\}$ has an infinite basic subsequence, we have (c) of Theorem I.

Suppose that (c) is not possible, then by Lemma 2 $\exists$ an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, that is $\overline{Y}$-overfilling with dimension of $\overline{Y} \geqslant 1$.

If $\overline{Y}$ is a finite codimensional subspace of $[y_n]$ by Lemma 4, $\exists$ an infinite subsequence $\{y_{n_k}\}$ of $\{y_n\}$, which is elementary of 1^{st} type, hence we have (a) of Theorem I.

Suppose now that

(41) $\{y_n\}$ is $\overline{Y}$-overfilling, with $\overline{Y}$ of dimension $p\geqslant 1$ and of infinite codimension in $[y_n]$.

By Theorem II* and (41) we have that $\exists\{h_n\}\subset B'$ and an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, so that

(42) $\{y_n, h_n\}$ is biorthogonal system, with $\{h_n\}\subset \overline{Y}^{\perp}(=\{f\in B';\ f(y)=0, \forall y\in\overline{Y}\})$.

Let us consider the sequence $\{y_n+\overline{Y}\}$ of $B/\overline{Y}$, by (41) and (D_{16}) this sequence is denucleated, hence by (12) $\exists$ an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, so that $\{y_n+\overline{Y}\}$ is basic. Therefore $\{y_n+\overline{Y}\}$ is uniformly minimal, consequently (**[9]**$_4$ theorem VI) $\{y_n+\overline{Y}\}$ has property P.

Then, if $y\in[y_n]$, $\exists$ two infinite complementary subsequences $\{n'_k\}$ and $\{n''_k\}$ of $\{n\}$ so that $y+\overline{Y}\in[y_{n'_k}+\overline{Y}]+[y_{n''_k}+\overline{Y}]$; that is $\exists y'\in[y_{n'_k}], y''\in[y_{n''_k}]$ and $\bar{y}\in\overline{Y}$ so that $y=y'+y''+\bar{y}$; but $\overline{Y}\subset[y_{n''_k}]$ by (41), that is $y''+\bar{y}\in[y_{n''_k}]$, hence $y\in[y_{n'_k}]+[y_{n''_k}]$, consequently

(43) $\{y_n\}$ has property P.

We have now two possibilities:

(A) Suppose that the dimension p of $\overline{Y}$ is >1.

Then, by (41) and by Lemma 3, $\{y_n\}$ has an infinite subsequence, which we call $\{y_n\}$ again, so that

(44) $\{y_n\}$ is convergent of order q to an I-sequence $\{\bar{y}_k\}_{k=1}^q$ of $\overline{Y}$, with $\bar{y}_m\notin[\bar{y}_{ms_n}]$, $\forall m$ and with codimension of $[\bar{y}_k]_{k=1}^q$ in $\overline{Y}\leqslant 1$.

Therefore we have two possible subcases:

(A_1) $\{\bar{y}_k\}_{k=1}^q$ is complete in $\overline{Y}$; then by (41), (42) and (44) we have the subcase (b_1) of (D_7); hence by (43) we have (b) of Theorem I.

(A_2) $[\bar{y}_k]_{k=1}^q$ has codimension 1 in $\overline{Y}$.

Then, in the Banach space $B/[\bar{y}_k]$, by (44) the sequence $\{y_n+[\bar{y}_k]\}$ is

$\overline{V}$-overfilling, with dimension of $\overline{V} = 1$. Hence by (D_{16}) $\{y_n + [\bar{y}_k]\}$ is without denucleated subsequences, therefore by (12) (see also [**1**] or [**8**]$_1$ p. 171) $\{y_n + [\bar{y}_k]\}$ has not an infinite basic subsequence; consequently ([**2**], see also [7] p. 128) $\exists$ an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, so that

(45) $\{(y_n + [\bar{y}_k])/\|y_n + [\bar{y}_k]\|\}$ is weakly convergent to $\bar{y} + [\bar{y}_k]$, with $\bar{y} + [\bar{y}_k] \neq 0$, hence $\bar{y} \cup \{\bar{y}_k\}_{k=1}^{q}$ is complete in $\overline{Y}$.

On the other hand, by (44) and by hypothesis of (A_2) $\{(y_n + [\bar{y}_k])/\|y_n + [\bar{y}_k]\|\}$ has not convergent subsequences, because $\{\bar{y}_k\}_{k=1}^{q}$ of (44) is the I-sequence of all the limit points; therefore, by (45) and by Theorem III*, $\exists\{\bar{\alpha}_n\} \subset \mathscr{C}$, $\{\tilde{y}_n\} \subset B$ and an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, so that

(46) $(y_n + [\bar{y}_k])/\|y_n + [\bar{y}_k]\| = \bar{\alpha}_n(\bar{y} + [\bar{y}_k]) + (\tilde{y}_n + [\bar{y}_k])$, $\forall n$, with $\lim_{n\to\infty} \bar{\alpha}_n = 1$; moreover $\{\tilde{y}_n + [\bar{y}_k]\}$ is basic, weakly convergent to 0 and with $\inf_n \|\tilde{y}_n + [\bar{y}_k]\| > 0$.

By (45) and (46) $y_n - \tilde{y}_n \in \overline{Y}$ $\forall n$; hence by (42) $\{\tilde{y}_n, h_n\}$ is a biorthogonal system; moreover by (46) $\bigcap_{m=1}^{\infty} [\tilde{y}_n + [\bar{y}_k]]_{n>m} = \{0\}$, that is $\bigcap_{m=1}^{\infty} [\tilde{y}_n]_{n>m} \subseteq [\bar{y}_k]$; therefore ([**9**]$_4$ theorem I) $\exists\{v_n\} \subset [\bar{y}_k]$ so that $\{\tilde{y}_n + v_n\}$ is M-basic; then, setting $\tilde{y}_n + v_n = y_n^*$, $\forall n$, $\exists$ an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, with $\{y_n^*\}$ basic ([**2**], see also [7] p. 128); on the other hand $\tilde{y}_n + [\bar{y}_k] = y_n^* + [\bar{y}_k]$, $\forall n$, consequently by (46) we have that

(47) $(y_n + [\bar{y}_k])/\|y_n + [\bar{y}_k]\| = (\bar{\alpha}_n\bar{y} + y_n^*) + [\bar{y}_k]$ $\forall n$, with $\{y_n^*\}$ basic; moreover $\{y_n^* + [\bar{y}_k]\}$ is basic and weakly convergent to 0, with $\inf_n \|y_n^* + [\bar{y}_k]\| > 0$.

Therefore, by (41), (42), (44), (45), (46) and (47), we have the subcase (b_3) of (D_7); hence, by (43), we have (b) of Theorem I.

(B) Suppose that the dimension p of $\overline{Y}$ is $= 1$.
We have again two possible subcases:

(B_1) $\{y_n/\|y_n\|\}$ has a convergent subsequence, hence by (41) and (42) we have the subcase (b_1) of (D_7), therefore by (43) we have (b) of Theorem I.

(B_2) $\{y_n/\|y_n\|\}$ has not convergent subsequences; on the other hand by (41) $\{y_n\}$ has not an infinite basic subsequence; consequently ([**2**], see also [7]

p. 128) $\{y_n/\|y_n\|\}$ has a subsequence weakly convergent to $\bar{y}\neq 0$; therefore, by Theorem III*, we have the subcase (b_2) of (D_7); hence by (43) we have (b) of Theorem I.

This completes the proof of Theorem I.

5 – *Let* $\{y_n\}$ *be a sequence of* B *infinitely convergent to an I-sequence* $\{\bar{y}_k\}$, *then:* $\Rightarrow \exists\{\varepsilon_n\}\subset R^+$ *so that,* $\forall\{x_n\}\subset B$ *with* $\|x_n - y_n\| < \varepsilon_n\ \forall n$, $\{x_n\}$ *is infinitely convergent to* $\{\bar{y}_k\}$.

Proof. By hypothesis $\forall m\ \exists\{v_{mn}\}\subset[\bar{y}_{ms_n}]$ so that

$$\lim_{n\to\infty}(y_n + v_{mn})/\|y_n + v_{mn}\| = \bar{y}_m\,. \tag{48}$$

Let us now set

$$\varepsilon_n = \min\{\varepsilon_{mn};\, 1\leqslant m\leqslant n\},\ \text{where}\ \varepsilon_{mn} = \|y_n + v_{mn}\|/n,\ \forall n\ \text{and}\ m\,. \tag{49}$$

By (49) $\{\varepsilon_n\}\subset R^+$, then let $\{x_n\}\subset B$ so that

$$\|x_n - y_n\| < \varepsilon_n \qquad \forall n\,. \tag{50}$$

By (49) and (50) it follows that

$$\frac{\|x_n - y_n\|}{\|y_n + v_{mn}\|} < \frac{\varepsilon_n}{\|y_n + v_{mn}\|} \leqslant \frac{\varepsilon_{mn}}{\|y_n + v_{mn}\|} = \frac{1}{n},\quad \forall n\ \text{and}\ m\ \text{with}\ m\leqslant n\,. \tag{51}$$

Moreover, setting $a_{mn} = \|y_n + v_{mn}\|/\|x_n + v_{mn}\|$, by (51) it follows that

$$1-\frac{1}{n}<1-\frac{\|x_n - y_n\|}{\|y_n + v_{mn}\|} = \frac{\|y_n + v_{mn}\| - \|x_n - y_n\|}{\|y_n + v_{mn}\|} \leqslant \frac{\|x_n + v_{mn}\|}{\|y_n + v_{mn}\|} =$$

$$= \frac{1}{a_{mn}} \leqslant \frac{\|y_n + v_{mn}\| + \|x_n - y_n\|}{\|y_n + v_{mn}\|} = 1 + \frac{\|x_n - y_n\|}{\|y_n + v_{mn}\|} < 1 + \frac{1}{n}\,,$$

$\forall\ n$ and m with $m\leqslant n$; therefore it is

$$\lim_{n\to\infty} a_{mn} = 1\,,\qquad \forall m\,. \tag{52}$$

On the other hand by (51) we have that

$$\lim_{n\to\infty} \|x_n - y_n\|/\|y_n + v_{mn}\| = 0\,, \qquad \forall m\,. \tag{53}$$

Consequently by (48), (52) and (53) it follows that

$$\lim_{n\to\infty} \frac{x_n + v_{mn}}{\|x_n + v_{mn}\|} = \lim_{n\to\infty} a_{mn} \frac{x_n + v_{mn}}{\|y_n + v_{mn}\|}$$

$$= \lim_{n\to\infty} a_{mn} \left(\frac{y_n + v_{mn}}{\|y_n + v_{mn}\|} + \frac{x_n - y_n}{\|y_n + v_{mn}\|} \right) = \bar{y}_m, \forall m\,.$$

That is $\{x_n\}$ is infinitely convergent to $\{\bar{y}_k\}$, which completes the proof of Lemma 5.

Proof of Theorem II. It follows by Lemma 5 and by Theorem I.

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Summary

Una successione $\{y_n\}$ si dice « overfilling » se ogni sua sottosuccessione infinita è completa in $[y_n]$. Ne sono stati costruiti esempi mediante le successioni infinitamente convergenti, ma era finora sconosciuta la struttura della generica successione overfilling. Si dimostra in questa Nota che gli esempi dati già esauriscono la categoria di tali successioni: precisamente, se $\{y_n\}$ è overfilling, ne esiste una sottosuccessione $\{y_{n_k}\}$ infinitamente convergente ad una successione $\{\bar{y}_k\}$ di punti limiti, con $\{\bar{y}_k\}$ completa in $[y_n]$.

Inoltre sono messi in evidenza i due tipi elementari di successioni infinite, mediante i quali sono costruite tutte le successioni prive di sottosuccessioni basiche. La nota termina con un esame della stabilità di tali successioni elementari, per successioni abbastanza « vicine ».

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