Riv. Mat. Univ. Parma (4) 6 (1980), 419-424

J. S. GUPTA and D. K. BHOLA (*)

Quadratic mean of entire functions of several complex variables (**)

1 - Let T be the set of mappings $f\colon C^n \to C$ (C is the complex plane and C^n is the cartesian product of n copies of C) such that the image under f of an element $z = (z_1, z_2, \dots, z_n)$ of C^n is

$$f(z_1, z_2, \dots, z_n) = \sum_{m_1, m_2, \dots, m_n \in N} a_{m_1, m_2, \dots, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}, \tag{1.1}$$

with $r_c^f = +\overline{\infty}$ (r_c^f is the polyradius of convergence of the multiple power series defining f and $+\overline{\infty} = (+\infty, \dots, +\infty)$); N is the set of natural numbers $0, 1, 2, \dots$, $\langle a_{m_1, \dots, m_n} | m_1, \dots, m_n \in N \rangle$ is a multiple sequence in C, and $z_r = x_r + iy_r$ for $r = 1, 2, \dots, n$, where $x_r, y_r \in R$ (R is the field of reals). Since the multiple power series defining f converges for each $z \in C^n$, f is an entire function of n complex variables. For simplicity, we shall take $n = 2$.

On a closed polydisc D: $|z_i| \leqslant r_i$, $i = 1, 2$, the quadratic mean function I_2 of an entire function $f \in T$ is defined as

$$I_2(r_1, r_2; f) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f(r_1 \exp[i\theta_1], r_2 \exp[i\theta_2])|^2 \, d\theta_1 \, d\theta_2 , \tag{1.2}$$

and some of its properties are studied in this paper.

2 - Theorem 1. *$I_2(r_1, r_2; f)$ is an increasing function of $r_1 r_2$ and $\log I_2(r_1, r_2; f)$ is a convex function of $\log(r_1 r_2)$.*

(*) Indirizzo: Math. Dept., University of Jammu, Jammu 180001, J & K, India.
(**) Ricevuto: 13-VI-1979.

Proof. In order to show that I_2 is an increasing function of $r_1 r_2$, we obtain the series representation of I_2. We have

$$I_2(r_1, r_2; f) = \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} |f(r_1 \exp[i\theta_1], r_2 \exp[i\theta_2])|^2 \, d\theta_1 \, d\theta_2 .$$

But

$$|f(z_1, z_2)|^2 = \sum_{m,n\in N} a_{m,n} r_1^m r_2^n \exp[i(m\theta_1 + n\theta_2) \sum_{p,q\in N} \overline{a_{p,q}} r_1^p r_2^q \exp[-i(p\theta_1 + q\theta_2)]$$

$$= \sum_{m,n\in N} |a_{m,n}|^2 r_1^{2m} r_2^{2n} + \sum_{m,n\neq p,q} a_{m,n} \overline{a_{p,q}} r_1^{m+p} r_2^{n+q} \exp[i((m-p)\theta_1 + (n-q)\theta_2)] ,$$

the series on the right-hand side being absolutely and uniformly convergent for $\theta_1, \theta_2 \in [0, 2\pi]$. Integrating termwise over the interval $[0, 2\pi]$ with respect to θ_1 and θ_2 we, therefore, get

$$\int_0^{2\pi}\int_0^{2\pi} |f(z_1, z_2)|^2 = 4\pi^2 \sum_{m,n\in N} |a_{m,n}|^2 r_1^{2m} r_2^{2n} .$$

Hence

$$I_2(r_1, r_2; f) = \sum_{m,n\in N} |a_{m,n}|^2 r_1^{2m} r_2^{2n} . \tag{2.1}$$

The fact that I_2 is a steadily increasing function of $r_1 r_2$ now readily follows from (2.1).

We now prove the convexity of $\log I_2$. In (2.1) putting $k = m, n$ and $r^k = r_1^m r_2^n$, we get $I_2(r, f) = \sum_{k\in N} |a_k|^2 r^{2k}$, from which the proof follows as in the case of one variable which is similar to the proof of theorem 5.41 in [**3**].

Theorem. 2. *If $I_2(r_1, r_2; f^{(k)})$ is the quadratic mean function of $(\partial/\partial z_k) \cdot f(z_1, z_2)$, $k = 1, 2$, then, for any r_i, $i = 1, 2$, there is a number $r_k^0(r_i, f) \in R_+$ (R_+ is the set of positive reals), $i \neq k$, such that*

$$I_2(r_1, r_2; f^{(k)}) \geqslant \frac{I_2(r_1, r_2; f)}{2^2} \left(\frac{\log I_2(r_1, r_2; f)}{r_k \log r_k}\right)^2 , \qquad \text{for } r_k \geqslant r_k^0(r_i, f) . \tag{2.2}$$

Proof. We prove (2.2) for $k = 1$, since the proof for $k = 2$ is similar.

Let $\xi_1 \in C$ be such that $|\xi_1| = r_1$. Then

$$(2.3)\qquad I_2(r_1, r_2; f^{(1)}) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \Big|\frac{\partial}{\partial \xi_1} f(\xi_1, z_2)\Big|^2 d\theta_1\, d\theta_2$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \Big|\lim_{\delta\to 0}\frac{f(\xi_1, z_2) - f(\xi_1 - \xi_1\delta, z_2)}{\xi_1\delta}\Big|^2 d\theta_1\, d\theta_2$$

$$\geqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \lim_{\delta\to 0}\Big(\frac{|f(\xi_1, z_2)| - |f(\xi_1 - \xi_1\delta, z_2)|}{|\xi_1\delta|}\Big)^2 d\theta_1\, d\theta_2$$

$$\geqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\lim_{\delta\to 0}\frac{|f(\xi_1, z_2)|^2 + |f(\xi_1 - \xi_1\delta, z_2)|^2 - 2|f(\xi_1, z_2)|\,|f(\xi_1 - \xi_1\delta, z_2)|}{r_1^2\delta^2}\, d\theta_1\, d\theta_2 .$$

But, from Schwarz's inequality, we have

$$(2.4)\qquad \int_0^{2\pi}\int_0^{2\pi} |f(\xi_1, z_2)|\,|f(\xi_1 - \xi_1\delta, z_2)|\, d\theta_1\, d\theta_2$$
$$\leqslant \Big\{\int_0^{2\pi}\int_0^{2\pi} |f(\xi_1, z_2)|^2 d\theta_1\, d\theta_2 \int_0^{2\pi}\int_0^{2\pi} |f(\xi_1 - \xi_1\delta, z_2)|^2 d\theta_1\, d\theta_2\Big\}^{\frac12} .$$

Using (2.4) in (2.3) we, therefore, get

$$I_2(r_1, r_2; f^{(1)}) \geqslant \lim_{\delta\to 0}\Big\{\frac{(I_2(r_1, r_2; f))^{\frac12} - (I_2(r_1 - r_1\delta, r_2; f))^{\frac12}}{r_1\delta}\Big\}^2 ,$$

since all the integrals are uniformly convergent so we can bring the limit outside the integrals. We set

$$(2.5)\qquad g(r_1, r_2; f) = \frac{\log I_2(r_1, r_2; f)}{\log r_1} .$$

Then, for any value of r_2, it follows, from Theorem 1, that $g(r_1, r_2; f)$ is an increasing function of r_1 for $r_1 \geqslant r_1^0(r_2, f)$. Therefore

$$I_2(r_1, r_2; f^{(1)}) \geqslant \lim_{\delta\to 0}\Big\{\frac{r_1^{g(r_1, r_2; f)/2} - (r_1 - r_1\delta)^{g(r_1 - r_1\delta,\, r_2;\, f)/2}}{r_1\delta}\Big\}^2 ,$$

$$\geqslant \lim_{\delta\to 0}\Big\{\frac{r_1^{g(r_1, r_2; f)/2} - (r_1 - r_1\delta)^{g(r_1, r_2; f)/2}}{r_1\delta}\Big\}^2 = \frac{r_1^{g(r_1, r_2; f)}}{r_1^2}\lim_{\delta\to 0}\Big\{\frac{1 - (1-\delta)^{g(r_1, r_2; f)/2}}{\delta}\Big\}^2$$

$$= \frac{r_1^{g(r_1, r_2; f)}}{r_1^2}\Big\{\frac{g(r_1\, r_2; f)}{2}\Big\}^2 = \frac{I_2(r_1, r_2; f)}{2^2}\Big\{\frac{\log I_2(r_1, r_2; f)}{r_1 \log r_1}\Big\}^2 ,$$

which proves the theorem.

3 – In this section we establish formulas, in terms of I_2, for the order and type of an entire function $f \in T$. The finite order ϱ of f is defined ([**1**], p. 219) as

$$\limsup_{r_1, r_2 \to +\infty} \frac{\log\log M(r_1, r_2; f)}{\log (r_1 r_2)} = \varrho . \tag{3.1}$$

The lower limit in (3.1) is called the lower order of f and is denoted by $\lambda \in R_+ \cup \{0\}$. In case $\varrho \in R_+$, the type $\tau \in R_+^* \cup \{0\}$ (R_+^* is the set of extended positive reals) of f is defined ([**1**], p. 223) as

$$\limsup_{r_1, r_2 \to +\infty} \frac{\log M(r_1, r_2; f)}{r_1^\varrho + r_2^\varrho} = \tau . \tag{3.2}$$

We shall call the lower limit in (3.2) the lower type of f and shall denote it by $\nu \in R_+^* \cup \{0\}$.

Theorem 3. *If $f \in T$ is an entire function of order $\varrho \in R_+ \cup \{0\}$ and lower order $\lambda \in R_+ \cup \{0\}$, then*

$$\lim_{r_1, r_2 \to +\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log I_2(r_1, r_2; f)}{\log (r_1 r_2)} = \begin{matrix}\varrho\\ \lambda\end{matrix} . \tag{3.3}$$

If, however, f is of order $\varrho \in R_+$, type $\tau \in R_+^* \cup \{0\}$ and lower type $\nu \in R_+^* \cup \{0\}$, then

$$\lim_{r_1, r_2 \to +\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log I_2(r_1, r_2; f)}{r_1^\varrho + r_2^\varrho} = \begin{matrix}2\tau\\ 2\nu\end{matrix} . \tag{3.4}$$

Proof. It follows, from the definition of I_2, that

$$I_2(r_1, r_2; f) \leqslant (M(r_1, r_2; f))^2 , \tag{3.5}$$

where M is the maximum modulus function of f on D. Also, it follows, from (2.1), that

$$I_2(r_1, r_2; f) \geqslant (\mu(r_1, r_2; f))^2 , \tag{3.6}$$

where μ is the maximum term in the double series defining f. From (3.5) and (3.6), we get

$$(\mu(r_1, r_2; f))^2 \leqslant I_2(r_1, r_2; f) \leqslant (M(r_1, r_2; f))^2 . \tag{3.7}$$

But ([1], p. 219) for functions of finite order ϱ, as $r_1, r_2 \to +\infty$,

$$\log \mu(r_1, r_2; f) \sim \log M(r_1, r_2; f) . \tag{3.8}$$

Hence, from (3.7) and (3.8), we have

$$\log (I_2(r_1, r_2; f))^{\frac{1}{2}} \sim \log M(r_1, r_2; f) , \tag{3.9}$$

as $r_1, r_2 \to +\infty$. The results in (3.3) and (3.4) now follow from (3.9) and the definition of ϱ, λ, τ, and ν.

Dzrbasyan, M. M. [2] has defined another order $\varrho_k \in R_+$ with respect to the variable z_k of an entire function $f \in T$ as

$$\limsup_{r_j \to +\infty} \limsup_{r_k \to +\infty} \frac{\log \log M(r_1, r_2; f)}{\log r_k} = \varrho_k , \tag{3.10}$$

where $j, k = 1, 2$ and $j \neq k$. We also establish two formulas for ϱ_k in terms of I_2 in the next theorem.

Theorem 4. *If $f \in T$ is an entire function of order $\varrho_k \in R_+$, then*

$$\begin{aligned} \limsup_{r_j \to +\infty} \limsup_{r_k \to +\infty} \frac{\log \log I_2(r_1, r_2; f)}{\log r_k} &= \varrho_k \\ &= \limsup_{r_j \to +\infty} \limsup_{r_k \to +\infty} \frac{\log \left(r_k I_2^{(k)}(r_1, r_2; f)/I_2(r_1, r_2; f)\right)}{\log r_k} , \end{aligned} \tag{3.11}$$

where $j, k = 1, 2$ and $j \neq k$, and $I_2^{(k)} = (\partial/\partial r_k) I_2(r_1, r_2; f)$.

Proof. The first equality in (3.11) follows from (3.9) and (3.10). We, therefore, prove the second equality in (3.11), but for $k = 1$, since the proof for $k = 2$ is similar. Since, in view of Theorem 1, $\log I_2$ is an increasing convex function of $\log r_1$, for some $r_2 \in R_+ \cup \{0\}$, we can write $\log I_2$ as

$$\begin{aligned} \log I_2(r_1, r_2; f) &= \log I_2(r_1^0, r_2; f) + \int_{r_1^0}^{r_1} \frac{(\partial/\partial x_1) I_2(x_1, r_2; f)}{I_2(x_1, r_2; f)} \, dx_1 \\ &\leq \log I_2(r_1^0, r_2; f) + r_1 \frac{I_2^{(1)}(r_1, r_2; f)}{I_2(r_1, r_2; f)} . \end{aligned}$$

Hence, in view of the first equality in (3.11),

$$\varrho_1 \leqslant \limsup_{r_2 \to +\infty} \limsup_{r_1 \to +\infty} \frac{\log \left(r_1 I_2^{(1)}(r_1, r_2; f) / I_2(r_1, r_2; f) \right)}{\log r_1}. \tag{3.12}$$

Also

$$\log I_2(2r_1, r_2; f) = \log I_2(r_1, r_2; f) + \int_{r_1}^{2r_1} \frac{(\partial/\partial x_1) I_2(x_1, r_2; f)}{I_2(x_1, r_2; f)} \mathrm{d}x_1 \geqslant r_1 \frac{I_2^{(1)}(r_1, r_2; f)}{I_2(r_1, r_2; f)}$$

and hence, again by the first equality in (3.11),

$$\varrho_1 \geqslant \limsup_{r_2 \to +\infty} \limsup_{r_1 \to +\infty} \frac{\log \left(r_1 (I_2^{(1)}(r_1, r_2; f) / I_2(r_1, r_2; f) \right)}{\log r_1}. \tag{3.13}$$

Combining (3.12) and (3.13), we get the desired result.

Corollary 1. *Under the hypotheses of Theorem* 4

$$\limsup_{r_j \to +\infty} \limsup_{r_k \to +\infty} \frac{\log \left(r_k (I_2(r_1, r_2; f^{(k)}) / I_2(r_1, r_2; f) \right)^{\frac{1}{2}}}{\log r_k} \geqslant \varrho_k. \tag{3.14}$$

The proof follows from (2.2) in view of (3.9) and (3.10).

References

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[3] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford 1939.

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