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CHEH-CHIH YEH (*)

Approximate iterative process in strictly convex Banach spaces (**)

Dedicated to Professor Kiyoshi Iseki on his 60th birthday

1 - Introduction

Let T be a selfmapping of a Banach space X. The mapping T is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all x, y in X. It is known [**7**] that the Picard sequence of iterates $\{T^n x_0\}$ for given x_0 in X, need not converges to a fixed point of T whereas the sequence $\{U^n x_0\}$ may converge to a fixed point of T, where $U = \lambda I + (1-\lambda) T$, $0 < \lambda < 1$. The study of the convergence of $\{U^n x_0\}$ was made in uniformly convex Banach space by Krasnoselski [**7**] for $\lambda = \frac{1}{2}$ and for a general λ by Schaeffer [**11**]. However, Edelstein [**3**], for $\lambda = \frac{1}{2}$ and Diaz and Metcalf [**2**] assumed the space only to be strictly convex. Recently, Massa $[\mathbf{9}]_1$ discussed the similar problem for $U = \sum_{i=0}^{\infty} c_i T^i$, where $c_i \geqslant 0$, $c_0 > 0$, $c_1 > 0$, $\sum_{i=0}^{\infty} c_i = 1$. Kannan [**5**] considered a selfmapping T of X satisfying $\|Tx - Ty\| \leqslant \frac{1}{2}\{\|x - Tx\| + \|y - Ty\|\}$.

Now let K be a closed and convex subset of a strictly convex Banach space X. A selfmapping T of K is called generalized nonexpansive if

(i) the set $F(T)$ of the fixed points of T is nonempty,

(*) Indirizzo: Dept. Math. National Central University, Chung-Li, Taiwan, Republic of China.

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(ii) $\|Tx - Ty\| \|x - y\| \leqslant \|x - y\| \{a_1(x, y)\|x - y\| + a_2(x, y)(\|x - Tx\| + \|y - Ty\|) + a_3(x, y)(\|x - Ty\| + \|y - Tx\|)\} + a_4(x, y)\|x - Ty\| \|y - Tx\|,$

for all x, y in K, $x \neq y$, where a_i: $K \times K \to [0, 1]$ with

$$\text{(C)} \qquad \begin{aligned} &\sup_{x,y \in K} \{a_1(x, y) + 2a_2(x, y) + 2a_3(x, y) + a_4(x, y)\} \leqslant 1 , \\ &\inf_{x,y \in K} \{a_1(x, y) + a_2(x, y) + a_3(x, y)\} > 0 . \end{aligned}$$

Let $S = \sum_{i=0}^{\infty} b_i T^i$, where b_i is constant for $i = 0, 1, 2, \ldots$, $b_i \geqslant 0$, $b_0 > 0$, $b_1 > 0$ and $\sum_{i=0}^{\infty} b_i = 1$.

In this present paper, we obtain results for approximately fixed points for this much wider class of mappings. We shall study the convergence of $\{S^n x\}$ for x in K and prove that S is asymptotically regular [**1**], i.e. $\lim_n \|S^{n+1}x - S^n x\| = 0$ for each x in K. Densifying mappings are also discussed.

For other related results, we refer to Kirk [**6**], Massa [**9**]$_2$ and Petryshyn and Williamson [**11**].

2 - Main results

Henceforth we suppose that $0 \in K$ and $T(0) = 0$.

Lemma 1. *Let $p \in F(T)$. Then $\|T^n x - p\| \leqslant \|T^{n-1}x - p\|$ for every $x \in K$ and $n \in N$, where N denotes the set of positive integers.*

Proof. Since $x \in F(T)$ is trivial, we can suppose that $x \in K - F(T)$. From (ii)

$$\begin{aligned} \|Tx - Tp\| \|x - p\| \leqslant \|x - p\| \{ &a_1(x, p)\|x - p\| + a_2(x, p)(\|x - Tx\| + \|p - Tp\|) \\ &+ a_3(x, p)(\|x - Tp\| + \|p - Tx\|)\} + a_4(x, p)\|x - Tp\| \|p - Tx\| \\ \leqslant \|x - p\| \{ &a_1(x, p)\|x - p\| + a_2(x, p)(\|x - p\| + \|p - Tx\|) \\ &+ a_3(x, p)(\|x - p\| + \|p - Tx\|)\} + a_4(x, p)\|x - p\| \|p - Tx\|, \end{aligned}$$

which implies

$$\big(1 - a_2(x, p) - a_3(x, p) - a_4(x, p)\big)\|Tx - p\|$$

$$\leqslant \big(a_1(x, p) + a_2(x, p) + a_3(x, p)\big)\|x - p\| .$$

Hence, by (C),

$$\|Tx - p\| \leqslant \frac{a_1(x, p) + a_2(x, p) + a_3(x, p)}{1 - a_2(x, p) - a_3(x, p) - a_4(x, p)}\|x - p\| \leqslant \|x - p\| .$$

Similarly, we can prove $\|T^n x - p\| \leqslant \|T^{n-1}x - p\|$. Thus $\|T^n x - p\| \leqslant \|x - p\|$ for every $n \in N$.

Remark. Since $0 \in F(T)$, $\|T^n x\| \leqslant \|x\|$.

Theorem 1. $F(T) = F(S)$.

Proof. If $x \in F(T)$, then $Sx = x$. Thus $x \in F(S)$. Hence we only prove « $x \in F(S) \Rightarrow x \in F(T)$ ». Let $x \in F(S)$, i.e., $x = \sum_{i=0}^{\infty} b_i T^i x$. If $b_1 = 1$, then $S = T$. Thus $F(S) = F(T)$. If $b_1 < 1$, then

$$\|x\| = \|b_1 Tx + \sum_{i\neq 1}^{\infty} b_i T^i x\| \leqslant b_1\|Tx\| + (1 - b_1)\|\sum_{i\neq 1}^{\infty} \frac{b_i}{1 - b_1} T^i x\|$$

$$\leqslant b_1\|Tx\| + \sum_{i\neq 1}^{\infty} b_i \|x\| = \|x\| .$$

Hence $\|x\| = \|Tx\|$ and $\|y\| = \|x\|$, where $y = \sum_{i\neq 1}^{\infty}\big(b_i/(1 - b_1)\big)T^i x$. Since X is strictly convex, we have $y = Tx$. Thus $x = Sx = b_1 Tx + (1 - b_1)y = Tx$. This completes our proof.

Lemma 2. *Let $p \in F(S)$ and $x \in K - F(S)$. Then* $\|Sx - p\| < \|x - p\|$.

Proof. If $b_0 = 1$, then $Sx = x$. Thus $x \in F(S)$, a contradiction. Hence $b_0 < 1$. Let $z = \sum_{i=1}^{\infty}\big(b_i/(1 - b_0)\big)T^i x$. If $z = x$, then $(1 - b_0)x = \sum_{i=1}^{\infty} b_i T^i x$. This means that $x = \sum_{i=0}^{\infty} b_i T^i x$. Thus $x \in F(S)$, a contradiction. Hence $z \neq x$. By

Theorem 1, $p \in F(S) = F(T)$. Hence

$$\|z-p\| = \|\sum_{i=1}^{\infty} \frac{b_i}{1-b_0} T^i x - \sum_{i=1}^{\infty} \frac{b_i}{1-b_0} p\| \leqslant \sum_{i=1}^{\infty} \frac{b_i}{1-b_0} \|T^i x - p\|$$
$$\leqslant \sum_{i=1}^{\infty} \frac{b_i}{1-b_0} \|x-p\| = \|x-p\| .$$

Hence

$$\|Sx-p\| = \|b_0(x-p) + (1-b_0)(z-p)\| < \|x-p\| .$$

Theorem 2. *Let T be continuous and $\{S^n x_0\}$ have a convergent subsequence for some $x_0 \in K$. Then $\{S^n x_0\}$ converges to a fixed point of T.*

Proof. Since T is continuous, S is continuous. Let $S^{n_i} x_0 \to y$ as $i \to \infty$. If $Sy \neq y$, then, from Lemma 2, the sequence $\{\|S^n x_0 - p\|\}$ is strictly decreasing. Since S is continuous, $S^{n_i+1} x_0 \to Sy$. Hence

$$\lim_i \|S^{n_i} x_0 - p\| = \|y-p\| > \|Sy-p\| = \lim_i \|S^{n_i+1} x_0 - p\| ,$$

a contradiction. Thus $Sy = y$.

Theorem 3. *If X is uniformly convex Banach space, then S is asymptotically regular.*

Proof. Let $p \in F(T) = F(S)$. For any point x_0 of K, we define $x_n = S^n x_0$ for $n \in N$. It follows from Lemma 1 that $\{\|x_n - p\|\}$ is decreasing and $\|x_n - p\| \to r$ for some $r \geqslant 0$. If $r = 0$, then $x_n \to p$ as $n \to \infty$. This implies $x_{n+1} - x_n = S^{n+1} x_0 - S^n x_0 \to 0$ as $n \to \infty$. Thus S is asymptotically regular at x_0. Suppose now that $r > 0$. Since $\|Sx_n - p\| = \|x_{n+1} - p\| \leqslant \|x_n - p\| \to r$ as $n \to \infty$, it follows from the uniform convexity of X that

$$\|x_n - p - (Sx_n - p)\| \to 0 ,$$

as $n \to \infty$. This implies $\|x_n - Sx_n\| = \|S^n x_0 - S^{n+1} x_0\| \to 0$ as $n \to \infty$. Thus our proof is complete.

The following definition is due to Furi and Vignoli [**4**].

Definition. A continuous selfmapping T of a metric space X is called *densifying* if for every bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(T(A))$

$< \alpha(A)$, where α denotes the measure of noncompactness of bounded sets in the sence of Kuratowski [**8**].

Theorem 4. *Let Y be a bounded closed convex subset of X and T a densifying generalized nonexpansive selfmapping of Y. Then $\{S^n x\}$ converges to a fixed point of T for each x in Y.*

Proof. From Theorem 1, $F(S) = F(T)$. The mapping S is defined on Y and $S(Y) \subset Y$. Let A be a bounded subset of Y with $\alpha(A) > 0$. Then

$$\alpha\big(S(A)\big) \leqslant \sum_{i=0}^{\infty} b_i \alpha\big(T^i(A)\big) < \sum_{i=0}^{\infty} b_i \alpha(A) = \alpha(A) .$$

Thus S is densifying.

Let $B = \bigcup_{n=0}^{\infty} S^n x$ for any x in Y. Then $S(B) \subset B$. Since S is continuous, $S(\bar{B}) \subset \overline{S(B)} \subset \bar{B}$.

To prove $\alpha(B) = 0$, suppose $\alpha(B) > 0$. Then we have

$$\alpha(B) = \max\{\alpha\big(S(B)\big), \alpha(x)\} = \alpha\big(S(B)\big) ,$$

a contradiction. Hence $\alpha(B) = 0$. Since X is a complete metric space, B is compact. If $x \in Y - F(S)$, then, by Lemma 2, $\|Sx - p\| < \|x - p\|$, for any $p \in F(T)$. As in the proof of Theorem 2, we can prove that $\{S^n x\}$ converges to a fixed point of T.

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