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A remark to the characterization of closed derivations in C^*-algebras (**)

1 - Introduction

Let A be a topological algebra [**2**]. A linear mapping δ in A is said to be a derivation if it satisfies the following conditions:

(1) the domain $D(\delta)$ is a topological subalgebra of A,

(2) $\delta(ab) = \delta(a)\,b + a\delta(b)$ for all $a, b \in D(\delta)$ (see [**1**], [**3**]).

Under the additional assumptions:

(3) A is a C^*-algebra,

(4) $D(\delta)$ is a dense *-subalgebra of A,

(5) $\delta(a^*) = \delta(a)^*$ for all $a \in D(\delta)$,

δ will be called a **-derivation in A* [**4**].

Sakai has noted the following problem ([**4**], problem 4), which is interesting for the study of C^*-differential manifolds.

Problem (Herman-Powers). Let $C[0, 1]$ and let δ be a closed derivation in $C[0, 1]$. Can we characterize δ? (For example, $\delta = f(x)(\mathrm{d}/\mathrm{d}x)$, where $f(x)$ is some function on $[0, 1]$).

We will give a partial answer.

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2 - Closed derivations

Let B be a commutative Banach-algebra. We define a derivation δ in B to be *a closed derivation in* B if

(6) from $a_n \in D(\delta)$, $a_n \to a$ and $\delta(a_n) \to b$ it follows $a \in D(\delta)$, $b = \delta(a)$ always ($\to$ stands for the convergence in B).

In [1] and [3] representation theorems for continuous derivations of rings and algebras are established. Here we will prove a theorem for derivations in Banach-algebras, which are not necessarily continuous derivations.

Theorem 1. *Suppose that B is a commutative Banach-algebra having the unit element $\boldsymbol{e}$ and δ_1 is a derivation in B with $e \in D(\delta_1)$ having the properties:*

(7) *there is an element $x \in D(\delta_1)$ such that either condition $\delta_1(x) = e$ or $\boldsymbol{e}/\delta_1(x) \in B$ is fulfilled;*

(8) *for any element $a \in D(\delta_1)$ there is a sequence $(p_n(x))$ of polynomials in x with scalar coefficients such that $p_n \to a$ and $\delta_1(p_n) \to \delta_1(a)$ (obviously the set of all polynomials of this kind belongs to $D(\delta_1)$).*

Now let δ be any closed derivation in B, whose domain $D(\delta)$ includes e and x. Then δ has the domain $D(\delta) = D(\delta_1)$ and the representation

$$\delta = b\delta_1, \tag{9}$$

where $b \in B$ depends on δ, or δ is an extension of the derivation $b\delta_1$, such that the restriction of δ to $D(\delta_1)$ has the form (9).

Proof. $D(\delta)$ is an algebra, which includes e and x. Therefore every polynomial $p(x) = \alpha_0\, e + \alpha_1 x + \ldots + \alpha_k x^k$ belongs to $D(\delta)$. From (2) it follows $\delta(x^n) = nx^{n-1}\delta(x)$ and $\delta_1(x^n) = nx^{n-1}\delta_1(x)$, hence $\delta(x^n) = f(x)\,\delta_1(x^n)$ $(n = 1, 2, \ldots)$, where $f(x) = \delta(x)$ or $f(x) = \delta(x)/\delta_1(x)$ (corresponding to (7)). Because of $\delta(e) = \delta_1(e) = 0$ (0 is the zero element in B) we obtain

$$\delta(p(x)) = f(x)\,\delta_1(p(x))\,, \tag{10}$$

for any polynomial in x with scalar coefficients. Now let a be any element in $D(\delta_1)$. For assumption (8) we can find a sequence $(p_n(x))$ of polynomials such that $p_n \to a$ and $\delta_1(p_n) \to \delta_1(a)$. By use of (10) we get

$$\delta(p_n) = f(x)\,\delta_1(p_n) \to f(x)\,\delta_1(a)\,.$$

On the other hand the derivation δ is closed, such that $a \in D(\delta)$ and $f(x)\,\delta_1(a) = \delta(a)$. This finished the proof.

It is well known that the linear space $C[0, 1]$ of all continuous functions $f(x)$ on $[0, 1]$ is a commutative Banach-algebra with unit element $e = g_1(x) \simeq 1$ and with zero divisors (under the topology of the uniform convergence on $[0, 1]$ and the pointwise multiplication). In fact, $C[0, 1]$ is a C^*-algebra. The derivation $\delta_1 = \mathrm{d}/\mathrm{d}x$ with $D(\delta_1) = C^1[0, 1]$ (the algebra of all continuously differentiable functions on $[0, 1]$) is a closed derivation in $C[0, 1]$. Obviously the function $x \in C^1[0, 1]$ fulfils (7). Now let $h(x)$ be any function in $C^1[0, 1]$, then $(\mathrm{d}/\mathrm{d}x)\, h(x) := h'(x) \in C[0, 1]$ can be approximated uniformly by polynomials, $p_n(x) \to h'(x)$. It is easy to see that $\int_0^x p_n(t)\,\mathrm{d}t \to \int_0^x h'(t)\,\mathrm{d}t$ holds too, hence $q_n(x) := h(0) + \int_0^x p_n(t)\,\mathrm{d}t \to h(x)$, where $q_n(x)$ are polynomials in x and $q_n'(x) \to h'(x)$. That means that (8) holds, too.

Therefore we have

Theorem 2. *Let δ be any closed derivation in $C[0, 1]$, whose domain $D(\delta)$ includes 1 and x, then $D(\delta) = C^1[0, 1]$ and*

$$\delta = f(x)(\mathrm{d}/\mathrm{d}x), \tag{11}$$

where $f(x) = \delta(x) \in C[0, 1]$, or δ is an extension of the derivation $f(x)(\mathrm{d}/\mathrm{d}x)$.

Remark. It is possible to define closed derivations δ in $C[0, 1]$ having the form $\delta = f(x)(\mathrm{d}/\mathrm{d}x)$, where $f(x) \notin C[0, 1]$ and $x \notin D(\delta)$. For example, the derivation $\delta = \big(1/h(x)\big)(\mathrm{d}/\mathrm{d}x)$, $h(x) \in C[0, 1]$, is a closed derivation in $C[0, 1]$ with a domain consisting of all functions $g(x) \in C^1[0, 1]$ which are so that $\big(1/h(x)\big)\, g'(x)$ can be made to a function in $C[0, 1]$. In fact if $\big(g_n(x)\big)$ is a sequence in $D(\delta)$ with $g_n(x) \to g(x) \in C[0, 1]$ and $\delta(g_n) \to \varphi(x) \in C[0, 1]$, then we have $h(x)\, \delta(g_n) = g_n'(x) \to h(x)\, \varphi(x)$, too. Since $\mathrm{d}/\mathrm{d}x$ is a closed derivation in $C[0, 1]$ we obtain $g(x) \in C^1[0, 1]$ and $g'(x) = h(x)\, \varphi(x)$ such that from $\big(1/h(x)\big)\, g'(x) = \varphi(x)$ it follows $g(x) \in D(\delta)$ and $\delta(g) = \big(1/h(x)\big)\, g'(x) = \varphi(x)$. Hence δ is a closed derivation.

An open question is the following: Can we characterize every closed derivation in $C[0, 1]$, whose domain does not include x, in such a form, where $f(x) \notin C[0, 1]$?

References

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