Riv. Mat. Univ. Parma (4) 6 (1980), 161-165

J. ACHARI and B. K. LAHIRI (*)

A fixed point theorem (**)

1 – The well-known Banach's fixed point theorem states that if T is a self mapping of a complete metric space M into itself such that

$$d(Tx, Ty) \leqslant \alpha \cdot d(x, y)\,, \tag{1}$$

for all $x, y \in M$ and $0 \leqslant \alpha < 1$, then T has a unique fixed point in M. A number of recent generalizations of this theorem may be seen in [**1**], [**2**], [**3**]$_{1,2}$, [**4**], [**6**], [**7**]$_1$, [**10**].

If T satisfies a more general condition

$$d(Tx, Ty) \leqslant a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)\,, \tag{2}$$

for all $x, y \in M$ where $a_i \geqslant 0$ and $\sum_{i=1}^{5} a_i < 1$, then condition (2) reduces to (1), if $a_i = 0$, $i = 2, \ldots, 5$; it reduces to a condition of Reich [**9**]$_{1,2,3}$ if $a_4 = a_5 = 0$ and to a condition of Kannan [7]$_1$ if $a_1 = a_4 = a_5 = 0$, $a_2 = a_3 = 1/2$.

If in (2), $\sum_{i=1}^{5} a_i$ is allowed to be equal to 1, then in a uniformly convex Banach space a fixed point theorem has been obtained in [**5**] which is an extension of a theorem of Soardi [**12**] and in a reflexive Banach space, a fixed point theorem has been obtained in [**2**] by assuming $\sum_{i=1}^{5} a_1 \leqslant 1$ which is an extension

---

(*) Indirizzo: Dept. Math., University of Kalyani, West Bengal, India.

(**) The paper was prepared while the authors were at Istituto Matematico « Ulisse Dini », Firenze; the first author as a fellow and the second author as a Visiting Professor under the invitation of the C.N.R. Administration, Italy. — Ricevuto- 6-XI-1978.

of a theorem of Kirk [**8**] who obtained the same result with $a_2 = a_3 = a_4 = a_5 = 0$.

Kirk [**8**] used the notion of normal structure to obtain his theorem while Kannan $[\mathbf{7}]_2$ limited the superior of the expression $\|y - Ty\|$ to prove his theorem. In this paper, we obtain a fixed point theorem in a reflexive Banach space taking advantages of both Kirk's notion of normal structure and Kannan's limitation of $\|y - Ty\|$. The method of proof of the theorem is patterned after the paper [**8**] with necessary modifications as required in the more general settings.

2 – Definition [**8**]. A bounded convex set K in a Banach space X is said to have normal structure if for each convex subset S of K which contains more than one point, there exists $x \in S$ such that $\sup_{y\in S} \|x - y\| < \delta(S)$, $\delta(S)$ being the diameter of S.

Theorem. *Let X be a reflexive Banach space and K be a non-empty closed convex bounded subset of X. Suppose that T is a mapping of K into itself such that*

(A) $\|Tx - Ty\| \leqslant \max\{\|x - y\|;\ \|x - Tx\|;\ \|y - Ty\|\}$ *for every* $x, y \in K$,

(B) *K has normal structure*,

(C) $\sup_{y\in F} \|y - Ty\| \leqslant \frac{1}{2}\delta(F)$ *for every non-empty closed convex subset F of K, containing more than one element and mapped into itself by T.*

Then T has a fixed point in K.

Proof. Let Y denote the family of all non-empty closed convex subsets of K which are mapped into itself by T. By Smulian's result [**11**], i.e. X is reflexive if, and only if, every decreasing sequence of non-empty bounded closed convex subsets of X has a non-empty intersection and by Zorn's lemma, it follows that Y has a minimal element F, say. If F contains only one element, then that element is a fixed point of T. We prove that F contains only one element. We suppose, therefore, that F contains more than one element and we obtain a contradiction.

Let $A = \sup_{y\in F} \|Ty - y\|$, then by the condition (C), $A \leqslant \frac{1}{2}\delta(F)$. Let further for $x \in F$,

$$\gamma_x(F) = \max\{(\sup \|x - y\| : y \in F);\ A\},$$

$$\gamma(F) = \inf\{\gamma_x(F) : x \in F,$$

$$F_c = \{x \in F : \gamma_x(F) = \gamma(F)\}.$$

We show that F_c is non-empty, closed and convex. For positive integer n and for $x \in F$, let

$$F(x, n) = \{y \in F \colon \|x - y\| \leqslant \gamma(F) + 1/n\}, \qquad C_n = \bigcap_{x \in F} F(x, n).$$

It is clear that C_n is a non-empty closed convex set and $C_{n+1} \subset C_n$. Moreover, because $\gamma(F) \geqslant A$ and $A \leqslant \frac{1}{2}\delta(F)$, we have

$$F_c = \bigcap_{n=1}^{\infty} C_n,$$

and so F_c is closed convex and by Smulian's result [**11**], non-empty.

Next we show that $\delta(F_c) < \delta(F)$. Since K has normal structure and $A \leqslant \frac{1}{2}\delta(F)$, there exists a point $x \in F$ such that $\gamma_x(F) < \delta(F)$. If x_1 and x_2 are any two points of F_c, then $\|x_1 - x_2\| \leqslant \gamma_{x_1}(F) = \gamma(F)$. So

$$\delta(F_c) = \sup\{\|x_1 - x_2\|;\ x_1, x_2 \in F_c\} \leqslant \gamma(F) \leqslant \gamma_x(F) < \delta(F).$$

Now, if $x \in F_c$ then for $y \in F$

$$\|Tx - Ty\| \leqslant \max\{\|x - y\|;\ \|Tx - x\|,\ \|Ty - y\|\} \leqslant \max\{\|x - y\|,\ \sup_{y \in F}\|Ty - y\|\}$$
$$\leqslant \max\{\|x - y\|,\ A\} \leqslant \gamma_x(F) = \gamma(F).$$

So, $T(F)$ is contained in a closed sphere $\overline{U}$, say with centre at Tx and radius $\gamma(F)$. Since $T(F \cap \overline{U}) \subset F \cap \overline{U}$ and F is minimal, it follows that $F \subset \overline{U}$ and so $\{\sup \|Tx - y\| : y \in F\} \leqslant \gamma(F)$.

Now

$$\gamma_{Tx}(F) = \max\{(\sup\|Tx - y\|;\ y \in F);\ A\}$$
$$\leqslant \max\{\gamma(F);\ A\} = \gamma(F), \qquad \text{because } A \leqslant \gamma(F).$$

Hence $\gamma_{Tx}(F) = \gamma(F)$. So, $Tx \in F_c$. Therefore F_c is a non-empty closed convex subset of F which is mapped into itself by T and because $\delta(F_c) < \delta(F)$, F_c is a proper subset of F. This contradicts the minimality of F. Hence F contains only one point which is a fixed point of T. This proves the theorem.

Recently, Ćirić [**3**]$_3$ obtained a fixed point theorem in a metric space by considering two more terms $\|x - Ty\|$ and $\|y - Tx\|$ within the bracket of the right hand side of (A), but he used a non-negative multiplier q which is *less than* 1 to the maximum of the terms $\|x - y\|$, $\|x - Tx\|$, $\|y - Ty\|$, $\|x - Ty\|$ and $\|y - Tx\|$.

We now exhibit some examples showing the different possibilities on the conditions (A) and (C), where in each case the norm is the usual norm of the real numbers.

Example 1. Let $K = [0, 1]$ and $Tx = 1 - x$. Here the mapping T satisfies the condition (A) for arbitrary $x, y \in [0, 1]$. The condition (C) is not satisfied.

Example 2. Let $K = [0, 1]$ and $Tx = (1/2)x + 1/4$. Here the conditions (A) and (C) are both satisfied. The example also shows that there may exist a pair of points (in this case $x = 0$, $y = 1/8$) x and y such that $\max\{|x - y|, |Tx - x|, |Ty - y|\}$ may be greater than $|x - y|$, where all the conditions of the theorem are satisfied.

Example 3. Let $K = [-1/4, 1]$ and $Tx = 1/2$ for $0 \leqslant x \leqslant 1$, $x \neq 1/2, 1/4$; $Tx = 8/9$ for $x = 1/2$; $Tx = -1/4$ for $x = 1/4$; $Tx = 1/4$ for $-1/4 \leqslant x < 0$. Here the condition (C) is satisfied, but for $x = 1/4$, $y = 1/2$, the condition (A) is not satisfied.

References

[1] V. W. Bryant, *A remark on a fixed point theorem for iterated mappings*, Amer. Math. Monthly, **75** (1968), 399-400.

[2] M. K. Chakraborty and B. K. Lahiri, *On fixed points of operators*, Indian J. Math. (2) **18** (1976), 81-89.

[3] Lj. Ćirić: $[\bullet]_1$ *Generalized contractions and fixed point theorem*, Publ. Inst. Math. (26) **12** (1971), 19-26; $[\bullet]_2$ *Fixed point theorems for mappings with a generalized contractive iterate at point*, Publ. Inst. Math. (27) **13** (1972), 11-16; $[\bullet]_3$ *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267-273.

[4] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7-10.

[5] K. Goebel, W. A. Kirk and T. N. Shimi, *A fixed point theorem in uniformly convex spaces*, Boll. Un. Mat. Ital. (4) **7** (1973), 67-75.

[6] L. F. Guseman, jr., *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. **26** (1970), 615-618.

[7] R. Kannan: $[\bullet]_1$ *Some results on fixed points* (II), Amer. Math. Monthly **76** (1969), 405-408; $[\bullet]_2$ *Some results on fixed points* (III), Fund. Math. **70** (1971), 169-177.

[8] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-1006.

[9] S. Reich: [•]$_1$ *Kannan's fixed point theorem*, Boll. Un. Mat. Ital. (4) **4** (1971), 1-11; [•]$_2$ *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121-124; [•]$_3$ *Fixed points of contractive functions*, Boll. Un. Mat. Ital. (4) **5** (1972), 26-42.

[10] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. **23** (1969), 631-634.

[11] V. Smulian, *On the principle of inclusion in the space of type* (B), Math. Sb. (N.S.) **5** (1939), 327-328.

[12] R. Soardi, *Su un problema di punto unito di S. Reich*, Boll. Un. Mat. Ital. (4) **4** (1971), 841-845.

Abstract

The paper contains a theorem on fixed point of operators that may be considered as an extension of a theorem of Kirk [**8**]. *Some examples have also been exhibited showing the different possibilities on the conditions of the theorem.*

* * *