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O. AKINYELE (*)

On the partial stability of the non-linear abstract Chauchy problem (**)

1 - Introduction

The technique of obtaining most results on stability or asymptotic stability consists of dividing the neighbourhood of some kind of invariant set or any other convenient set into suitable subsets and then showing that either the solutions cannot leave such sets or estimate the escape time. This idea suggests that an efficient method of generalizing stability results is to establish some general results in terms of arbitrary sets in Euclidean spaces [**3**], [**4**]$_2$, [**5**] or Banach spaces [**1**]$_{1,2}$ and apply then to study the various problems of stability and boundedness of differential equations in such spaces.

The theory of partial differential equations as is well-known play a key role in many fields of mathematical applications, for example, control process, waves, etc. The problem of Lyapunov stability and asymptotic behaviour of solutions is of particular interest. On the other hand, certain classes of partial differential equations can be formulated as operator differential equations in a suitable Banach space and it turns out to be advantageous to handle such operator equations with a view to transfering results so obtained to the original partial differential equations. It is therefore of considerable interest to study stability properties of differential equations in Banach spaces.

In the last 15 years, Lyapunov stability results have been refined and further generalized in several directions. One such generalization is the concept of partial stability of differential equations in Euclidean spaces, which has been

(*) Indirizzo: Department of Mathematics, University of Ibadan, Nigeria.
(**) Ricevuto: 5-IX-1978.

studied by several authors [2]$_{1,2}$, [9], [10]. This type of stability is useful from the practical point of view since in certain situations one may be interested in the behaviour of only some of the variables of the solution.

In this paper, we wish to introduce the concept of partial stability of differential equations in an infinite dimensional Banach space, and obtain sufficient conditions for such concepts to hold. Our results are naturally in the framework of the Lyapunov function theory and the comparison technique. We approach the problem by first obtaining global results of general character, which generalize our earlier results in [1]$_{1,2}$ and then applying these results to obtain sufficient conditions for the partial equistability, partial uniform stability, partial uniform asymptotic stability and partial boundedness of solutions of the non-linear abstract Cauchy problem.

2 - Main results

We shall introduce and study the concept of partial stability for the non-linear abstract Cauchy problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + F(t, z)\,, \qquad z(t_0) = z_0 \in D[A(t_0)]\,, \tag{1}$$

where $z \in Z$ is an infinite dimensional Banach space, $R^+ = [0, \infty)$ and $F \in C(R^+ \times Z, Z)$; the Banach space of continuous functions from $R^+ \times Z$ into Z. For each $t \in R^+$, $A(t)$ is a closed linear operator in Z, with dense time-varying domain $D[A(t)]$ and generally unbounded. We shall assume the existence of solutions of the system (1) for all $t \geqslant t_0$.

Consider along side (1) the scalar differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t, x)\,, \qquad x(t_0) = x_0\,, \tag{2}$$

where $g \in C(R^+ \times R, R)$; the space of continuous functions from $R^+ \times R$ into $R = (-\infty, \infty)$, and let $x(t, t_0, x_0)$ denote any solution of (2) through (t_0, x_0) while $z(t, t_0, z_0)$ is any solution of (1) through (t_0, z_0).

In what follows, we take W to be a closed subspace of Z and P to be the projection operator from Z onto W.
We also denote by $\bar{Q}$, ∂Q and PQ, the closure, the boundary and the projection of Q onto W respectively for any set $Q \subset Z$.

In the following theorem we present a general set of sufficient conditions which prevents the solutions of (1) which has its origin in a given set $Q \subset Z$ from passing through any given part of the boundary ∂PQ.

Theorem 2.1. *Assume that*

(i) $V \in C(R^+ \times E, R)$ *and* $V(t, z)$ *is locally Lipschizian, in* z;

(ii) *for a Banach subspace* $W \subset Z$, *there exists sets* $I_1, I_2 \subset R^+$, $Q \subset Z$ *and* $G \subset W$ *such that* $I_1 \cap I_2 \neq \emptyset$, $\bar{Q} \subset E$ *and* $G \subset \partial PQ$;

(iii) $a \in C(R^+, R)$ *and* $V(t, z) \geq a(t)$ *for* $(t, z) \in R^+ \times H$, *where* $H = \{z \in E: Pz \in G\}$;

(iv) $z_0 \in Q$, $t_0 \in I_1$ *and* $V(t_0, z_0) < a(t_0)$;

(v) $g \in C(R^+ \times R, R)$ *and for* $(t, z) \in R^+ \times J$, *where* $J = \{z \in E: Pz \in PQ\}$, $D^+ V(t, z) = \limsup_{h \to 0^+} (1/h)[V(t + h, z + hF(t, z)) - V(t, z)] \leq g(t, V(t, z))$;

(vi) *for each* $t \in R^+$ *and all* $h > 0$ (h *sufficiently small*), *the operator* $R[h, A(t)] = [I - hA(t)]^{-1}$ *exists as a bounded operator defined on* Z *and for each* $z \in Z$, $\lim_{h \to 0} R[h, A(t)]z = z$;

(vii) *any solution* $x(t, t_0, x_0)$ *of the scalar differential equation* (2) *satisfies the inequality* $x(t, t_0, x_0) < a(t)$, $t \geq t_0$, *provided* $x_0 < a(t_0)$ *and* $t_0 \in I_2$.

Then there exists no $t^* > t_0$, $t_0 \in I = I_1 \cap I_2$ *such that* $z(t, t_0, z_0) \in J$ *for* $t \in [t_0, t^*)$ *and* $z(t^*, t_0, z_0) \in H$.

Proof. Assume that $\exists t^* > t_0$ such that $t_0 \in I$, $z(t, t_0, z_0) \in J$ for $t \in [t_0, t^*)$ and $z(t^*, t_0, z_0) \in H$. Then hypothesis (iii) implies $V(t^*, z(t^*, t_0, z_0) \geq a(t^*)$. Let $x_0 = V(t_0, z_0)$ for $t_0 \in I$, then $z(t, t_0, z_0) \in J$ for $t \in [t_0, t^*)$ together with (v), (vi) and arguments parallel to those in theorem 2.2 of [**1**]$_1$ implies

$$V(t, z(t, t_0, z_0)) \leq r(t, t_0, x_0) \quad \text{for } t \in [t_0, t^*),$$

where $r(t, t_0, x_0)$ is the maximal solution of (2). Letting $x_0 \in Q$, and using (iv), (vii) along with arguments similar to the concluding part of the proof of theorem 2.2 of [**1**]$_1$, we arrive at a contradiction and the result follows.

Remark. Theorem 2.1 contains corollary 2.3 of [**1**]$_1$ as a special case if we take $I_1 = I_2 = R^+$ and $W = Z$, so that the projection operator P becomes the identity operator. We also note that if we use vector Lyapunov functions in Theorem 2.1 instead of scalar Lyapunov functions, the proofs can be constructed with obvious modifications and we thus obtain a natural generalization of Theorem 2.2 of [**1**]$_1$. This result also generalizes a theorem of Ladde and Leela [**3**] obtained in the case $Z = R^n$.

The next result gives a set of sufficient conditions required for the solutions of (1) which start in a given set to reach another given set in a finite time and remain there for all future time.

Theorem 2.2. *Assume that*

(i) $V \in C(R^+ \times E, R)$ *and* $V(t, z)$ *is locally Lipschitzian in* z;

(ii) *there exist sets* $A \subset E$, $Q_0 \subset E$, $I_1 \subset R^+$ *such that* $z_0 \in A$, $t_0 \in I_1$ *implies that* $z(t, t_0, z_0) \in B$ *for* $t \geqslant t_0$ *where* $B = \{z \in E,: Pz \in PQ_0\}$;

(iii) $g \in C(R^+ \times R, R)$ *and for* $(t, z) \in R^+ \times B$, $D^+ V(t, z) \leqslant g(t, V(t, z))$;

(iv) *there exists a set* $D \subset Q_0$ *such that* $\bar{D} \subset Q_0$ *and* $V(t, z) \geqslant a(t)$ *for* $(t, z) \in R^+ \times D_0$, *where* $a \in C(R^+, R)$ *and* $D_0 = \{z \in E: Pz \in P(Q_0 \sim D)\}$;

(v) *for each* $t \in R^+$ *and all* $h > 0$ *(h sufficiently small) the operator* $R[h, A(t)] = [I - hA(t)]^{-1}$ *exists as a bounded operator defined on* Z *and for each* $z \in Z$: $\lim_{h \to 0} R[h; A(t)]z = z$;

(vi) *there exists a set* $I_2 \subset R^+$ *such that* $I = I_1 \cap I_2 \neq \emptyset$ *and a number* $T_0 = T_0(t_0, x_0) > 0$, $t_0 \in I_2$, $x_0 > 0$ *such that for any solution* $x(t, t_0, x_0)$ *of* (2) *the relation,* $x(t, t_0, x_0) < a(t)$, $t \geqslant t_0 + T_0$, $t_0 \in I_2$ *holds, provided* $x_0 < a(t_0)$.

Then there exists a real number $T = T(t_0, z_0)$ *such that* $z_0 \in A$, $t_0 \in I = I_1 \cap I_2$ *implies that* $Pz(t, t_0, z_0) \in PD$ *for* $t \geqslant t_0 + T$.

Proof. Let $z_0 \in A$ and $t_0 \in I$, then by (ii) $z(t, t_0, z_0) \in B$ for $t \geqslant t_0$. Set $x_0 = V(t_0, z_0)$, $t_0 \in I$ and $T = T(t_0, z_0) = T_0(t_0, V(t_0, z_0))$. By (i), (iii) and (v) together with arguments parallel to that of theorem 2.1 of $[\mathbf{1}]_2$ we obtain, $V(t, z(t, t_0, z_0)) \leqslant r(t, t_0, x_0)$, $t \geqslant t_0$, where $r(t, t_0, x_0)$ is the maximal solution of (2) through (t_0, x_0).

Let $\{t_k\}$ be a sequence such that $t_k \geqslant t_0 + T$, $t_0 \in I$ and $t_k \to \infty$ as $k \to \infty$ and assume that $z(t_k, t_0, z_0) \in D_0$ for $t_k \geqslant t_0 + T$. Proceding as in the concluding part of Theorem 2.1 leads to a contradiction and hence the results.

Remark. If we take $I_1 = I_2 = R^+$ and $W = Z$, then the projection operator P becomes the identity operator and so theorem 2.2 of $[\mathbf{1}]_2$ is included in the last Theorem. It is well-known (cfr. $[\mathbf{4}]_1$) that the use of vector Lyapunov functions is advantageous in certain situations. If in Theorem 2.2 we use vector Lyapunov functions instead of scalar Lyapunov functions our result generalizes theorem 2.1 of $[\mathbf{1}]_2$. Our result is also a generalization of a similar result of [**3**] obtained in the case $Z = R^n$.

3 - Application of main results

We shall introduce in this section refined concepts of stability for the abstract Cauchy problem (1). In particular, we shall apply our results in section **2** to cover several situations of partial stability and boundedness criteria of solutions of (1).

Let X and Y be infinite dimensional Banach spaces and let $Z = X \times Y$ in (1) with the norm of Z defined as follows

$$\|(u, v)\|_Z = \|u\|_X + \|v\|_Y .$$

Clearly with this norm Z becomes an infinite dimensional Banach space, and the abstract Cauchy problem (1) is therefore equivalent to the system

$$(3) \qquad \frac{du}{dt} = A(t)u + f(t, u, v) , \qquad \frac{dv}{dt} = A(t)v + g(t, u, v) ,$$

where $u \in X$, $v \in Y$, $f \in C(R^+ \times Z, X)$, $g \in C(R^+ \times Z, Y)$ and $F \in C(R^+ \times Z, Z)$. We shall assume conditions on F which guarantee the existence of solution of (1).

Definition 3.1. (i) The set $z = 0$ is said to be *partially equistable or equi-stable with respect to the u-component* if given $\varepsilon > 0$ $\exists \delta(t_0, \varepsilon) > 0$ such that $\|u_0\|_X + \|v_0\|_Y < \delta$ implies $\|u(t, t_0, z_0)\|_X < \varepsilon$ for $t \geqslant t_0$.

(ii) The set $z = 0$ is said to be *partially uniformly stable* or *uniformly stable with respect to the u-component* if δ in (i) is independent of t_0.

(iii) The set $z = 0$ is said to be *partially asymptotically stable* if for each $\varepsilon > 0$, $t_0 \in R^+$ there exists positive numbers $\delta = \delta(t_0)$ and $T = T(t_0, \varepsilon)$ such that for $t \geqslant t_0 + T$ and $\|u_0\|_X + \|v_0\|_Y < \delta_0$, $\|u(t, t_0, z_0)\|_X < \varepsilon$.

(iv) The set $z = 0$ is said to be *partially uniformly asymptotically stable* if δ and T in (iii) are independent of t_0 and (ii) also holds.

Definition 3.2. (i) The solutions of (1) are said to be *partially equi-bounded* if for each $\alpha \geqslant 0$, and $t_0 \in R^+$ there exists a positive function $\beta = \beta(t_0, \alpha)$ which is continuous in t_0 for each α such that $\|u_0\|_X + \|v_0\|_Y \leqslant \alpha$ implies $\|u(t, t_0, z_0)\|_X < \beta$ for $t \geqslant t_0$.

(ii) The solutions of (1) are said to be *partially uniformly bounded* if the β in (i) is independent of t_0.

Other partial boundedness properties could be formulated similar to the corresponding definitions of stability as in Definition 3.1. We omit details.

We now state sets of sufficient conditions for the partial stability and boundedness properties of Def. 3.1 and 3.2 to hold for the system (1). Denote by $S_\varrho(u)$ the set $\{u \in X \colon \|u\|_X < \varrho\}$.

Theorem 3.3. *Assume that*

(i) $V \in C(R^+ \times S_\varrho(u) \sim \{0\} \times Y, R)$, *$V(t, u, v)$ is locally Lipschitzian in (u, v) and $V(t, u, v) \to -\infty$ as $\|u\|_X + \|v\|_Y \to 0$, and $t \to \infty$;*

(ii) $b \in C([0, \varrho], R)$ *and for* $(t, u, v) \in R^+ \times S_\varrho(u) \sim \{0\} \times Y$, $V(t, u, v) \geq b(\|u\|_X)$;

(iii) $g \in C(R^+ \times R, R)$ *and for* $(t, u, v) \in R^+ \times S_\varrho(u) \sim \{0\} \times Y$, $D^+ V(t, u, v) = \limsup_{h \to 0^+} (1/h)[V(t + h, R[h, A(t)]u + hf(t, u, v), R[h, A(t)]v + hg(t, u, v)) - V(t, u, v)] \leq g(t, V(t, u, v))$;

(iv) *for each* $t \in R^+$ *and* $h > 0$, *the operator* $R[h, A(t)]$ *exists as a bounded operator on* $Z = X \times Y$ *and, for each* $(u, v) \in X \times Y$, $\lim_{h \to 0^+} (R[h, A(t)]u, R[h, A(t)]v) = (u, v)$;

(v) *for every* $r \in (0, \varrho)$ *there exists a* $\tau(r) > 0$ *such that any solution* $x(t, t_0, x)$ *of* (2) *satisfies* $x(t, t_0, x_0) < b(r)$ *for* $t \geq t_0 \geq \tau(r)$ *provided* $x_0 < b(r)$.

Then the set $z = 0$ *is partially uniformly stable.*

Proof. Let $0 < \varepsilon_0 < \varrho$, then there exists $\delta(\varepsilon_0) > 0$ and $\tau(\varepsilon_0) > 0$ such that $(u_0, v_0) \in S_\varrho(u) \sim \{0\} \times Y$, $t_0 \geq \tau(\varepsilon_0)$ implies $V(t_0, u_0, v_0) < b(\varepsilon_0)$ by assumption (i). Now set $E = S_\varrho(u) \sim \{0\} \times Y$, $Q = \{(u, v): \|u\|_X + \|v\|_Y < \varepsilon_0\} \sim \{0\}$, $G = P\partial Q = \{u \in X: \|u\|_X = \varepsilon_0\}$ and $W = X$. In addition set $a(t) = b(\varepsilon_0)$ and $I_1 = I_2 = [\tau(\varepsilon_0), \infty)$. With these choices, the hypothesis of Theorem 2.1 are all satisfied and hence the uniform partial stability of the set $z = 0$ follows.

Remark. Assume that the set $z = 0$ is self-invariant with respect to (1) and suppose we assume that $V(t, u, v) \to -\infty$ as $\|u\|_X + \|v\|_Y \to 0$ for each $t \in R^+$, with $\tau(r) = 0$ for every r such that $0 < r < \varrho$, then following the arguments of the above theorem we have the partial equistability of the set $z = 0$ with respect to the problem (1). On the other hand if we assume in Theorem 3.3 that the set $z = 0$ is an asymptotically self-invariant set (see [4]$_1$ for definition of asymptotically self-invariant set), then the stability of the asymptotically self-invariant set $z = 0$ is also called the eventual stability of the set ([4]$_1$ 4.7), and so the conclusion of Theorem 3.3 in that case implies the partial eventual uniform stability of the set $z = 0$ with respect to the Cauchy problem (1).

Let $M_\varrho = \{u \in X: \|u\|_X \geq \varrho\}$, we now give a set of sufficient conditions for the partial uniform boundeness of solutions of (1).

Theorem 3.4. *Assume that*

(i) $V \in C(R^+ \times M_\varrho \times Y, R)$, $V(t, u, v)$ *is locally Lipschitzian in* u, v;

(ii) $b \in C([\varrho, \infty), R)$ *and for* $(t, u, v) \in R^+ \times M_\varrho \times Y$, $V(t, u, v) \geq b(\|u\|_X)$;

(iii) *for every* $(t, r) \in R^+ \times (\varrho, \infty)$, *there exists* $\beta(r) > \varrho$ *such that* $\|u\|_X + \|v\|_Y \leq r$ *implies* $V(t, u, v) < b(\beta(t))$;

(iv) $g \in C(R^+ \times R, R)$ *and for* $(t, u, v) \in R^+ \times M_\varrho \times Y$, $D^+ V(t, u, v) \leq g(t, V(t, u, v))$;

(v) *for each* $t \in R^+$, $h > 0$, *the operator* $R[h, A(t)]$ *satisfies condition* (iv) *of Theorem* 3.3;

(vi) *for every* $0 < r < \varrho$, $\exists \tau(r) > 0$ *such that any solution* $x(t, t_0, x_0)$ *of* (2) *satisfies* $x(t, t_0, x_0) < b(\beta(r))$ *for* $t \geqslant t_0 \geqslant \tau(r)$ *provided* $x_0 < b(\beta(r))$.

Then the solutions of (1) *are partially uniformly bounded.*

Proof. Let $\alpha \geqslant \varrho$ and $t_0 \in R^+$, then $\exists \beta(\alpha) > \varrho$ such that $\|u_0\|_X + \|v_0\|_Y \leqslant \alpha$ implies $V(t_0, u_0, v_0) < b(\beta(\alpha))$. Setting $E = M_\varrho \times Y$, $Q = \{(u, v): \|u\|_X + \|v\|_Y < \beta\}$ and $G = P\partial Q = \{u \in X: \|u\|_X = \beta\}$, $W = X$ and $a(t) = b(\beta(\alpha))$, then we see clearly that $\|u_0\|_X + \|v_0\|_Y \leqslant \alpha$ implies $\|u(t, t_0, z_0)\|_X < \beta$ for all $t \geqslant t_0$, using Theorem 2.1.

Theorem 3.5. *Assume that in addition to the hypothesis of Theorem* 3.3, (i) $b(s)$ *is non-decreasing in* s *and for every* $0 < r < \varrho$, *there exists a* $\tau = \tau(r) > 0$ *and* $T(r) > 0$ *such that every solution* $x(t, t_0, x_0)$ *of* (2) *satisfies* $x(t, t_0, x_0) < b(r)$ *for* $t \geqslant t_0 + T$, *with* $t_0 \geqslant \tau$. *Then the set* $z = 0$ *is partially uniformly asymptotically stable.*

Proof. By Theorem 3.3, the set $z = 0$ is partially uniformly stable, hence for $\varepsilon_0 = \varrho$, there exists $\delta_0 = \delta(\varrho) > 0$ such that $\|u_0\|_X + \|v_0\|_Y < \delta$ implies that $u(t, t_0, z_0) \in S_\varrho(u)$ for $t \geqslant t_0$. Now in Theorem 2.2, set $A = S_{\delta_0}(u)$, $Q_0 = E = S_\varrho(u) \sim \{0\} \times Y$, $D = S_{\varepsilon_1}(u) \times Y$ for any $\varepsilon_1 \in (0, \varrho)$ and $a(t) = b(\varepsilon_1)$. We see that conditions (ii) and (iv) of Theorem 2.2 are satisfied. Other hypothesis of the theorem are also clearly satisfied, hence an application of the theorem implies that if $\|u_0\|_X + \|v_0\|_Y < \delta$, then $(u(t, t_0, z_0), v(t, t_0, z_0)) \in D$ which implies that $u(t, t_0, z_0) \in S_{\varepsilon_1}(u)$ for $t \geqslant t_0 + T$, thus establishing the partial asymptotic stability of the set $z = 0$.

Remark. We can formulate sufficient conditions for the other types of partial stability and boundedness properties of the set $z = 0$ of the abstract Cauchy problem (1). All that is required to establish such results reduces to making appropriate choices of certain invariant sets parallel to the proofs of our results. We leave details of such formulations since they are fairly straightforward and parallel to our results here in this paper.

References

[1] O. Akinyele: [•]$_1$ *Stability of motions of non-linear abstract Cauchy problem*, Boll. Un. Mat. Ital. (5) **13**-A (1976), 322-331; [•]$_2$ *Asymptotic stability for the non-linear abstract Cauchy problem*, Riv. Mat. Univ. Parma (4) **4** (1978), 115-122.

[2] C. CORDUNEANU: [•]$_1$ *Sur la stabilité partielle,* Rev. Roumaine Math. Pures Appl. **9** (1964), 229-236; [•]$_2$ *Some problems concerning partial stability,* Symposia Math. **6** (1971), 141-154.

[3] G. LADDE and S. LEELA, *Global results and asymptotically self-invariant set,* Atti Accad. Naz. Lincei, Rend. (8) **54** (1973), 322-327.

[4] V. LAKSHMIKANTHAM and S. LEELA: [•]$_1$ *Differential and integral inequalities,* vol. I, Academic Press, New York 1969; [•]$_2$ *Global results and stability of motion,* Proc. Camb. Phil. Soc. **70** (1971), 95-102.

[5] N. ROUCHE and DANG-CAN PHIEN, *Sufficient stability conditions based on global lemmas,* Proc. Intern. Symposium on Non-linear oscillations, Kiev U.S.S.R. 1969.

[6] N. ROUCHE and KI PEIFFER, *Lyapunov's second method applied to partial stability,* J. Mécanique **8** (1969), 323-334.

[7] V. V. RUMIANTSEV, *On the stability of motion relative to a part of the variables,* Vestnik Moskov Univ. Ser. I **4** (1957), 9-16.

Riassunto

L'autore dà condizioni sufficienti per parziale stabilità e limitatezza delle soluzioni del problema di Cauchy.

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