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On von Neuman regular rings (IV) (**)

Introduction

In $[\mathbf{6}]_{1,2,3}$ the von Neumann regularity of rings is considered essentially through p-injectivity (for rings without identity, cfr. [5]). In $[\mathbf{6}]_4$, the regularity of rings whose left ideals are quasi-injective is considered and a few characteristic properties of regular rings are given in terms of annihilators. In this note, we study certain connections between regular rings, left V-rings and rings whose cyclic singular left modules are semi-simple. Among the results proved are the following: (1) If every cyclic singular left A-module is semi-simple, then A is a left V-ring iff every complement of any minimal left ideal of A is a maximal left ideal. (2) If every left ideal of A is two-sided, the following are equivalent: (a) A is regular; (b) any proper left ideal of A which contains every minimal projective left ideal is an intersection of maximal left ideals; (c) A satisfies the following conditions: (i) every cyclic singular left A-module is semisimple; (ii) every minimal left ideal of A is flat and $r(b) = l(b)$ for any $b \in A$. (3) A is a regular ring in each of the following cases: (a) A is a semi-prime, P. I-ring or left semi-Artinian ring whose cyclic singular left modules are semi-simple; (b) every one-sided essential ideal of A is an ideal and every factor ring of A is semi-primitive. (4) The group ring $A[G]$ is regular if A is a ring whose essential right ideals are ideals and G is a group such that $A[G]$ is a left V-ring. Partial answers are also given to the following questions raised by Fisher ([2], problems 1 and 3): (1) If each factor ring of A is semi-primitive and each primitive factor ring of A is regular, then is A regular? (2) Are prime left V-rings primitive?

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Throughout, A denotes an associative ring with identity and modules are unitary A-modules. We recall that: (1) The singular submodule of a left A-module M is $Z(M) = \{z \in M / l(z)$ is essential in ${}_A A\}$; M is called singular (non-singular) if $Z(M) = M (Z(M) = 0)$. (2) A left A-module M is semi-simple if the intersection of all maximal submodules of M is zero [**4**]. Then A is a left V-ring iff every left A-module is semi-simple ([**4**], theorem 2.1). Write: (1) A is a CSS-ring if every cyclic singular left A-module is semi-simple (such rings are neither regular nor left V-rings); (2) A is ELT (ERT) if every essential left (right) ideal of A is an ideal. It may be noted that any factor ring of an ERT-ring is ERT.

1 – The first two theorems are motivated by [**4**] (theorem 2.1).

Theorem 1. *The following conditions are equivalent.*

(i) *Every simple left A-module is either injective or projective,*

(ii) *Any proper left ideal of A which contains every minimal projective left ideal of A is an intersection of maximal left ideals.*

(iii) *Every minimal left ideal is projective and every singular left A-module is semi-simple.*

(iv) *A is a* CSS-*ring whose minimal left ideals are projective.*

Proof. (i) implies (ii). Let I be a proper left ideal of A containing every minimal projective left ideal. Then $M = A/I$ contains no simple projective submodule. For any $0 \neq y \in M$, by Zorn's Lemma ,the set of submodules of M not containing y has a maximal member Q. If T is the intersection of all submodules D of M with $Q \subset D$, then $y \in T$ and T/Q is simple. Since T/Q cannot be projective, then T/Q is injective and $M/Q = (T/Q) \oplus (U/Q)$ which implies $y \notin U$. Thus $Q = U$ is a maximal submodule of M which proves M semi-simple.

(ii) implies (i). Let S be a simple, non-projective left A-module, L a proper essential left ideal of A and $g: L \to S$ a non-zero left A-homomorphism. Then with $G = \ker g$, $L/G \approx S$ and if $G \cap R = 0$ for some minimal projective left ideal R of A, since $R \subseteq L$, $L = G \oplus R$ which yields $S \approx R$ projective, a contradiction. Thus G is an intersection of maximal left ideals of A and since L is also an intersection of maximal left ideals, there exists a maximal left ideal J such that $G \subseteq J$ but $L \nsubseteq J$. Since L/G is simple, $J \cap L = G$, and as $J + L = A$, g can therefore be extended to $h = A \to S$ which proves S injective.

(i) implies (iii). Let M be a singular left A-module and $0 \neq y \in M$. The set of submodules of M not containing y has a maximal member K. The intersection of all submodules of M which strictly contains K is $Ay + K$ and $(Ay + K)/K$ is a simple essential submodule of M/K. Since $y \in Z(M)$, $(Ay + K)/K$ is singular and therefore injective which implies $(Ay + K)/K = M/K$. Thus K is a maximal submodule of M and $y \notin K$ implies M semi-simple.

(iii) implies (iv) trivially.

(iv) will imply (i) if the proof of « (ii) implies (i) » is modified as follows: if G is not essential in ${}_AL$, there exists a minimal left ideal P such that $G \oplus P = L$ and $S \approx P$ is projective which is a contradiction. Thus G is essential in ${}_AA$ and is therefore an intersection of maximal left ideals.

Corollary 1.1. *If every simple left A-module is either injective or projective, then any left ideal containing the left socle of A is an intersection of maximal left ideals.*

Corollary 1.2. *A is a left V-ring iff A is a semi-prime* CSS*-ring such that every primitive factor ring of A is a left V-ring.*

(Apply [**6**]$_3$, proposition 6 and [**2**], theorem 14).

Corollary 1.3. *Let A be a semi-prime,* CSS*-ring and G a group. Then the group ring $A[G]$ is fully left idempotent iff G is locally finite and the order of any element in G is a unit in A.* (cfr. [**2**], theorem 9.)

Corollary 1.4. (i) *If A is a semi-prime,* CSS, P.I.*-ring, then A is a regular, left and right V-ring.* (ii) *A semi-prime,* CSS, *left semi-Artinian ring is regular.* (iii) *If A is a* P.I.*-ring and G a finite group, then $A[G]$ is a regular, left and right V-ring iff A is a semi-prime,* CSS*-ring and the order of G is a unit in A.*

(Apply [**2**], theorems 16 and 17, [**4**], corollaries 6.6 and 6.7 and [**6**]$_3$, proposition 6).

It is well-known that A is regular iff every left (right) A-module is flat. If A is fully left idempotent, then A/T is a flat right A-module for any ideal T of A ([**4**], lemma 2.3). Therefore [**1**] (proposition 2.1) and [**6**]$_3$ (proposition 6) yield.

Corollary 1.5. *A semi-prime,* ERT, CSS*-ring is a regular ring whose simple right modules are either injective or projective.*

[**1**] (proposition 2.1), [**3**] (theorem 3.9), [**4**], (lemma 3.1) and [**6**]$_3$ (proposition 6) imply

Corollary 1.6. *A semi-prime left Goldie ring whose cyclic singular left modules are either semi-simple or flat is a finite direct sum of simple rings.*

Corollary 1.7. *If A is an* ERT-*ring, G a group, $A[G]$ a left V-ring, then $A[G]$ is regular.*

(Apply [2], theorems 5 and 10).

Lemma 2.1. *Let A be a* CSS-*ring whose minimal left ideals are flat. Then every simple left A-module is either injective or flat. If, further, $l(b) = r(b)$ for any $b \in A$, then A is fully left and right idempotent.*

Proof. The validity of the first part follows from the proof of Theorem 1. Suppose now that $l(b) = r(b)$ for any $b \in A$. If $AbA + l(b) \neq A$, let L be a maximal left ideal containing $AbA + l(b)$. If A/L is injective, we have a contradiction as in the proof of [**6**]$_3$ (lemma 1). If A/L is flat, then by [**1**] (proposition 2.1), $b = bc$ for some $c \in L$ which implies $1 - c \in r(b) = l(b)$ and hence $1 \in L$, again a contradiction. Thus $AbA + l(b) = AbA + r(b) = A$ which yields $b \in (Ab)^2$ and $b \in (bA)^2$. This proves A fully left and right idempotent.

Now [**5**]$_2$ (theorem 5), [**6**]$_3$ (theorem 2), Theorem 1 and Lemma 2.1 imply

Proposition 2.2. *The following conditions are equivalent for a ring A whose left ideals are ideals.*

(i) *A is regular.*

(ii) *Any proper left ideal of A which contains every minimal projective left ideal is an intersection of maximal left ideals.*

(iii) *A is a* CSS-*ring whose minimal left ideals are flat and $r(b) = l(b)$ for any $b \in A$.*

As usual, a complement of a left ideal I of A is a left ideal K which is maximal with respect to $K \cap I = 0$.

Theorem 3. *The following conditions are equivalent.*

(i) *A is a left V-ring.*

(ii) *A is a* CSS-*ring such that every complement of any minimal left ideal of A is a maximal left ideal of A.*

Proof. (i) implies (ii). Let K be a complement of a minimal left ideal I of A. Since K and $L = K \oplus I$ are intersections of maximal left ideals ([**4**], theorem 2.1), there exists a maximal left ideal U such that $K \subseteq U$ but $L \not\subseteq U$. Then $U \cap I = 0$ which implies $K = U$.

(ii) implies (i). Let S be a simple left A-module, L a proper essential left ideal of A and $f: L \to S$ a nonzero left A-homomorphism. Then $F = \ker f$ is a maximal left subideal of L. If $F \cap I = 0$ for some non-zero left subideal I of L, then $L = F \oplus I$ and $I \approx S$. Let K be maximal with respect to $F \subseteq K$ and $K \cap I = 0$. Then, by hypothesis, $A = K \oplus I$ which shows that f may be extended to $g: A \to S$. Otherwise, F is essential in ${}_AL$ which implies both A/F and A/L are cyclic singular and there exists a maximal left ideal V of A such that $F \subseteq V$ but $L \nsubseteq V$. Then $A = L + V$ and $L \cap V = F$ which shows that f may again be extended to $h: A \to S$. This proves S injective.

The proof of Theorem 3 yields the following

Corollary 3.1. *If A has zero left socle, then A is a left V-ring iff A is a* CSS-*ring.*

Corollary 3.2. *If A has zero left socle and is a finitely generated module over its centre, then A is regular iff A is a* CSS-*ring.* (cfr. [**4**], corollary 6.4).

Corollary 3.3. *A semi-prime,* CSS-*ring such that every primitive factor ring has zero left socle is a left V-ring.*
(Apply Corollary 1.2).

Corollary 3.4. *If A has zero left socle and G is a finite group whose order is a unit in A, then $A[G]$ is a left V-ring iff A is a* CSS-*ring.* (cfr. [**4**], corollary 6.7).

Corollary 3.5. *A is a left V-ring iff A is a semiprime* CSS-*ring such that the complements of minimal left ideals of every primitive factor ring of A are maximal left ideals* (cfr. [**2**], theorem 14).

The next result is related to the following question raised by Fisher ([**2**], problem 3): Are prime left V-rings primitive?

Proposition 4. *Let A be a prime* CSS-*ring. Then either A is primitive or A is a left V-ring. If, further, A is either* ELT *or* ERT, *then A is primitive.*

Proof. If A contains a minimal left ideal, then A prime implies A primitive. If not, A is a left V-ring by Corollary 3.1. Now suppose A is ELT. If A has zero left socle, then any maximal left ideal L is essential in ${}_AA$ and A/L is an injective left A-module by Theorem 1. A is therefore strongly regular (cfr. the proof of ([**6**]$_1$, proposition 3)) which implies A is a field. Thus A is primitive in this case. Finally suppose A is ERT. Then A/R is a flat right

A-module for any essential right ideal R of A ([**4**], lemma 2.3) which implies A regular and the proof of [**6**]$_1$ (proposition 3) again implies A primitive.

Corollary 4.1. *Let A be a left V-ring such that every prime factor ring of A is* ERT. *Then A is a regular ring whose prime factor rings are primitive.* (cfr. [**2**], theorem 13).

Proposition 4 has the following analogue for regular rings.

Proposition 5. *A prime* ELT *regular ring is primitive.* (This is related to a problem of Kaplansky [**2**], p. 114).

Proposition 4 also yields the following partial answer to another question of Fisher ([**2**], problem 1) (cfr. Introduction and also [**2**], problem 4).

Proposition 6. *If A is an* ERT-*ring such that* (a) *every factor ring of A is semi-primitive and* (b) *every primitive factor ring of A is regular, then A is regular.*

Proof. For any essential right ideal R of A, A/R semiprimitive implies R is an intersection of maximal right ideals of A. Since any factor ring of an ERT-ring is ERT, by Theorem 1, the simple right modules of any prime factor ring F of A are either injective or projective. Then Proposition 4 implies F primitive. Thus every prime factor ring of A is regular which proves A regular (cfr. [**2**], p. 114).

[**1**] (proposition 2.1), [**4**] (lemma 2.3), [**6**]$_3$ (proposition 3), Theorem 1 and the proof of Proposition 6 show the validity of the next result.

Proposition 7. *A ring whose one-sided essential ideals are ideals and such that every factor ring is semi-primitive is regular.*

It is known that A is semi-simple, Artinian iff every semi-simple left A-module is injective ([**4**], theorem 3.2). We prove

Proposition 8. *If every semi-simple left A-module is either injective or projective, then A is left hereditary.*

Proof. By Theorem 1, every singular left A-module is semi-simple which implies every singular left A-module is either injective or projective. Let Q be an injective left A-module, M a submodule of Q. Then Q contains an injective hull E of M and $Q = E \oplus T$. Since E/M is singular, then E/M projective implies $M = E$. Thus E/M is injective and since $(M \oplus T)/M \approx T$ is injective, then $Q/M = (E/M) \oplus (M \oplus T)/M$ is injective which proves A left hereditary.

Corollary 8.1 *If every semi-simple left A-module is either flat injective or projective, then A is regular, left hereditary.*

For completeness, recall that a left A-module M is p-injective if, for any principal left ideal I of A and any left A-homomorphism $g: I \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in \mathrm{I}$ [**6**]. Our last result contains a generalisation of [**4**] (theorem 3.2).

Theorem 9. *The following conditions are equivalent.*

(i) *A is semi-simple, Artinian.*

(ii) *A is a semi-prime* ELT, CSS-*ring of left finite Goldie dimension.*

(iii) *A is a semi-prime* CSS, *left Goldie ring whose indecomposable injective left modules with the same associated prime ideal of A are isomorphic.*

(iv) *For every cyclic semi-simple left A-module M which is either singular or non-singular, either M is injective or M is p-injective with its injective hull projective.*

Proof. (i) implies (ii) through (iv) obviously.

(ii) implies (i). If I is a proper essential left ideal of A, L a maximal left ideal containing I, then A/L is injective by Theorem 1. Now A semi-prime ELT implies A left non-singular and hence A is a left Goldie ring. By [**3**] (theorem 3.9), L contains a non-zero-divisor c. If $f: Ac \to A/L$ is defined by $f(ac) = a + L$ for all $a \in A$, there exists $d \in A$ such that $1 + L = f(c) = cd + L$ which implies $1 \in L$ (two-sided), a contradiction. Thus the only essential left ideal of A is A which is therefore semi-simple, Artinian.

(iii) implies (i) as in the proof of [**4**] (theorem 3.2).

(iv) implies (i). Since every simple left A-module is p-injective, then every principal left ideal of A is semi-simple (cfr. $[\mathbf{6}]_3$, theorem 9). For any $z \in Z(A)$, Az p-injective implies Az a direct summand of ${}_AA$ and since $Z(A)$ contains no non-zero idempotent, then $z = 0$ which proves $Z(A) = 0$. Then every principal left ideal, which is p-injective, is a direct summand of ${}_AA$ which proves A regular ($[\mathbf{6}]_1$, lemma 2). If M is a cyclic left A-module with an injective hull E projective, then by a well-known lemma of Kaplansky, M is a direct summand of E which implies $M = E$. Then A is a left V-ring which implies every left A-module semi-simple [**4**] (theorem 2.1). Also A is a left self-injective ring. Now for any cyclic left A-module C, $C = Z(C) \oplus D$ by [**7**] (corollary 10) and by hypothesis, both $Z(C)$ and D are injective which proves C injective. Thus A is semi-simple, Artinian by [**4**] (theorem 3.2).

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Résumé

Dans cette note, on considère les anneaux tels que les modules singuliers cycliques soient semi-simples par rapport aux anneaux réguliers et les V-anneaux. On donne aussi des solutions partielles aux problèmes suivants de J. W. Fisher: (1) Peut-on caractériser un anneau régulier A par les propriétés suivantes: tout anneau quotient de A est semi-primitif et tout anneau quotient primitif de A est régulier? (2) Les V-anneaux premiers sont-ils primitifs?

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