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A note on countable compactifications (**)

1 - Introduction

Let X be a completely regular, T_1-space. We say that a Hausdorff compactification αX of X is *a countable compactification* if the remainder $\alpha X - X$ is countably infinite. The question of characterizing when X has a countable compactification has been answered for the locally compact case by Magill in [**4**]. For $\alpha X = \beta X$, where βX is the Stone-Čech compactification, Okuyama [**5**] has provided a characterization of when βX is a countable compactification. Apparently, little is known in case X is an arbitrary completely regular space. Magill's results show that a locally compact X has a countable compactification if and only if $\beta X - X$ contains infinitely many components. In [**3**] it is shown that the latter condition is not characteristic of when X possesses a countable compactification when X is not locally compact.

In this paper we prove that X has a countable compactification if and only if $\beta X - X$ is a countable, disjoint union of compact sets subject to certain convergence conditions in $\beta X - X$.

2 - Characterization of countable compactifications

All spaces considered herein are Hausdorff spaces. Let N denote the natural numbers and let $R = \mathrm{Cl}_{\beta X}(\beta X - X) \cap X$. Then R is precisely the set of points in X which do not possess compact neighborhoods. Evidently, $R = \mathrm{Cl}_{\alpha X}(\alpha X - X) \cap X$, for any compactification αX of X (see [**6**]).

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Suppose $\mathscr{S} = \{S_n \mid n \in N\}$ and $S = \cup \{S_n \mid n \in N\}$. If $\{x_\alpha\}$ and $\{y_\alpha\}$ are nets in S we say that $\{x_\alpha\}$ and $\{y_\alpha\}$ satisfy condition $(*)$ with respect to $\mathscr{S}$ if and only if for each index α, there exists $n(\alpha) \in N$ such that $x_\alpha \in S_{n(\alpha)}$ and $y_\alpha \in S_{n(\alpha)}$.

Theorem 1. *Let X be a completely regular, T_1-space. Then the following are equivalent.*

(A) *X has a countable compactification.*

(B) *$\beta X - X$ is a countable, disjoint union of compact sets K_n, $n \in N$. If $\mathscr{K} = \{K_n \mid n \in N\}$ and $\{x_\alpha\}$ and $\{y_\alpha\}$ are nets in $\beta X - X$ which satisfy condition $(*)$ with respect to $\mathscr{K}$, then:* (i) *$\{x_\alpha\}$ converges to a point $x_0 \in R$ implies $\{y_\alpha\}$ clusters at x_0;* (ii) *$\{x_\alpha\}$ converges to x_0, where $x_0 \in K_m$, implies $\{y_\alpha\}$ clusters at some point of K_m.*

Proof. (A) *implies* (B). Let αX be a countable compactification of X. Let π be the projection of βX onto αX and set $\alpha X - X = \{a_n \mid n \in N\}$. Take $K_n = \Pi^{-1}(a_n)$, for all $n \in N$. Since Π carries $\beta X - X$ onto $\alpha X - X$ (see Theorem 6.12 [**2**]), each K_n is compact and $\beta X - X$ is the (pair-wise) disjoint union of the sets K_n.

Let $\{x_\alpha\}$ and $\{y_\alpha\}$ be nets in $\beta X - X$ which satisfy condition $(*)$ w.r. to $\mathscr{K} = \{K_n \mid n \in N\}$. Suppose $\{x_\alpha\}$ converges to a point x_0 and $x_0 \in R$. Then $\{y_\alpha\}$ has a cluster point y_0 in $\beta X - (X - R)$. Select a subnet $\{y_{\alpha_\gamma}\}$ of $\{y_\alpha\}$ which converges to y_0. Since Π is continuous $\{\Pi(x_{\alpha_\gamma})\}$ converges to $\Pi(x_0)$ and $\{\Pi(y_{\alpha_\gamma})\}$ converges to $\Pi(y_0)$. But $\Pi(x_{\alpha_\gamma}) = \Pi(y_{\alpha_\gamma})$, for all γ, hence $\Pi(x_0) = \Pi(y_0)$. Since no point of βX distinct from x_0 is identified with x_0 under Π, it follows that $y_0 = x_0$.

Next, suppose that $x_0 \in K_m$, for some $m \in N$. Let $\{y_{\alpha_\gamma}\}$ be a subnet of $\{y_\alpha\}$ which converges to some $y_0 \in \beta X - (X - R)$. Now $\Pi(x_{\alpha_\gamma}) = \Pi(y_{\alpha_\gamma})$, for all γ, and it follows that $\Pi(x_0) = \Pi(y_0)$. Thus, $y_0 \in \Pi^{-1}\pi(x_0) = K_m$.

(B) *implies* (A). Let αX be the space obtained by identifying each K_n to a point a_n and leaving X invariant. Let Π be the associated projection of βX onto αX. Evidently, each fibre of Π is compact and Π is a continuous surjection whose restriction to X is an embedding of X in αX. To see that αX is Hausdorff we show that Π is a closed mapping.

Let F be closed in βX and consider $\Pi^{-1}(\Pi(F))$. Now $F = (F \cap X) \cup (F \cap \beta X - X)$ so that $\Pi^{-1}(\Pi(F)) = \Pi^{-1}[(\Pi(X \cap F) \cup \Pi(F \cap \beta X - X)] = F \cup \Pi^{-1}(\Pi(F \cap \beta X - X))$. Set $S = \Pi^{-1}(\Pi(F \cap \beta X - X))$. Let x_0 be a cluster point of $\Pi^{-1}(\Pi(F))$. If x_0 is a cluster point of F, then $x_0 \in F$ so that $x_0 \in \Pi^{-1}(\Pi(F))$. Otherwise, x_0 is a cluster point of S. Let $\{x_\alpha\}$ be a net in S which converges to x_0.

Case (i). $x_0 \in R$. Now there exists $n(\alpha)$ such that $x_\alpha \in K_{n(\alpha)}$, for each α. Since $\Pi(S) = \Pi(F \cap \beta X - X)$, there exists a net $\{y_\alpha\}$ in $F \cap \beta X - X$ such that $\Pi(y_\alpha) = \Pi(x_\alpha)$, for all α. Then $\{x_\alpha\}$ and $\{y_\alpha\}$ satisfy condition $(*)$ so that by B(i), $\{y_\alpha\}$ clusters at x_0. Since $\{y_\alpha\}$ is a net in F, $x_0 \in \Pi^{-1}(\Pi(F))$.

Case (ii). $x_0 \in K_m$, for some $m \in N$. As in case (i) choose $\{y_\alpha\}$ in $F \cap \beta X - X$ such that $\Pi(x_\alpha) = \Pi(y_\alpha) = a_{n(\alpha)}$, for all α. By B(ii), $\{y_\alpha\}$ clusters at a point $y_0 \in K_m$. Since $\{y_\alpha\}$ is a net in F, $y_0 \in F$. Thus $\Pi(y_0) \in \Pi(F)$, so that $K_m = \Pi^{-1}(\Pi(y_0)) \subseteq \Pi^{-1}(\Pi(F))$. Hence $x_0 \in \Pi^{-1}(\Pi(F))$ and $\Pi^{-1}(\Pi(F))$ is closed. Now Π is a perfect map so that αX is a countable Hausdorff compactification of X.

This completes the proof.

With appropriate modifications of the proof of Theorem 1, it can be seen that if R is compact and if each point of R has a countable base of neighborhoods in X, then the conditions concerning nets in Theorem 1 can be replaced by conditions involving sequences. Thus, we obtain

Theorem 2. *Let X be a completely regular, T_1 space with R compact. If each point of R has a countable base of neighborhoods in X, then the following are equivalent.*

(A) *X has a countable compactification.*

(B) *$\beta X - X$ is a countable, disjoint union of compact sets K_n, $n \in N$, and if $\{x_n\}$ and $\{y_n\}$ are sequences which satisfy $(*)$ relative to $\mathscr{K} = \{K_j \mid j \in N\}$, then*: (i) *$\{x_n\}$ converges to a point $x_0 \in R$ implies that x_0 is a cluster point of $\{y_n\}$*; (ii) *if all cluster points of $\{x_n\}$ lie in some K_m, then $\{y_n\}$ has a cluster point in K_m.*

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