Riv. Mat. Univ. Parma (4) 4 (1978), 7-13

V. P. MADAN and C. ROGERS (*)

Correspondence principle for plane strain problems in magneto-thermoviscoelasticity (**)

1 - Introduction

The use of correspondence principles in linear viscoelasticity is well known. The elastic-viscoelastic analogy which forms the basis of the correspondence principle, was first introduced by Alfrey [**1**] in determining stresses produced by external forces in a viscoelastic body. Lee [**5**] then extended the same, to deal with viscoelastic compressible bodies. Later, Hilton [**4**] generalized Alfrey's analogy to thermal stresses and Sternberg [**10**] introduced it into the theory of thermal stresses in compressible bodies. The treatment of such problems in viscoelasticity and thermoviscoelasticity may be found in the works of Bland [**2**] and Nowacki [**7**], respectively.

In the present work, we seek the solution of plane strain problems in magneto-thermoviscoelasticity by employing a correspondence principle. The solution of associated problem in magneto-thermoelasticity may be obtained without recourse to linearization, (as in the works of Paria [**8**], Madan [**6**] and Chandrasekharaih [**3**]).

2 - Magneto-thermoelastic problem

The governing equations of linear magneto-thermoelasticity comprise:

— A linear elastic stress-strain law involving the temperature distribution

$$\tau_{ij} = 2\mu_0 e_{ij} + (\lambda_0 e - \beta_0 T)\delta_{ij}, \qquad (1.1)$$

(*) Indirizzi: V. P. MADAN, Dept. of Math. Red Deer, College, Red Deer, Alberta, Canada. C. ROGERS, Dept. of Applied Math., University of Waterloo, Waterloo, Ont., Canada.

(**) Ricevuto: 30-I-1978.

in which λ_0, μ_0 are Lamé constants, $\beta_0 = (3\lambda_0 + 2\mu_0)\alpha$ where α is the coefficient of linear thermal expansion and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, u_i $(i = 1, 2, 3)$ represent the displacement components.

— Maxwell's electromagnetic equations

$$\text{(1.2)} \qquad \operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \qquad \text{(1.3)} \qquad \operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

$$\text{(1.4)} \qquad \operatorname{div} \boldsymbol{B} = 0, \qquad \text{(1.5)} \qquad \operatorname{div} \boldsymbol{D} = \varrho_e,$$

$$\text{(1.5)} \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \text{(1.2)} \qquad \boldsymbol{B} = \mu_e \boldsymbol{H}.$$

— Ohm's law

$$\boldsymbol{J} = \sigma \left[\boldsymbol{E} + \left(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\right)\right] + \varrho_e \frac{\partial \boldsymbol{u}}{\partial t} - k_0 \boldsymbol{\nabla} T_0,$$

where ϱ_e denotes thc charge donsity and σ the charge conductivity and k_0 is a constant.

— Fourier's law of heat conduction

$$\text{(1.9)} \qquad k \nabla^2 T_0 + Q = \varrho C_v \frac{\partial T_0}{\partial t} + T_1 \beta_0 \frac{\partial e}{\partial t} + \pi_0 \operatorname{div} \boldsymbol{J},$$

where Q represents the intensity of heat source, k is the thermal conductivity, C_v is the specific heat at constant strain, T_1 is a certain reference temperature over which the perturbed temperature is T_0 and π_0 is the coefficient connecting the current density with the heat flow density.

— The equations of motion for an electrically conducting elastic solid [7]

$$\text{(1.10)} \qquad \varrho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ik}}{\partial x_k} + (\boldsymbol{J} \times \boldsymbol{B})_i + \varrho_e \boldsymbol{E} + \boldsymbol{F}.$$

— In addition to the stresses τ_{ij} due to the elastic deformation the application of the electro-magnetic field also produces stresses in the medium and the corresponding stress tensor $\bar{\tau}_{ij}$ is called the Maxwell electro-magnetic stress. It is given in terms of the electric and magnetic field by

$$\text{(1.11)} \qquad \bar{\tau}_{ij} = \varepsilon \left[E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}\right] + \frac{1}{\mu_e} \left[B_i B_j - \frac{1}{2} B_k B_k \delta_{ij}\right]$$

— The total stress T_{ij} is defined by

$$T_{ij} = \tau_{ij} + \bar{\tau}_{ij} . \tag{1.12}$$

Equations (1.1)-(1.10) comprise the basic equations of magneto-thermo-elasticity. These are solved, using the prescribed initial and boundary conditions.

Neglecting the displacement vector $\boldsymbol{D}$, and external body force $\boldsymbol{F}$ and assuming static situations, the eqs. (1.1), (1.10) for finite conductivity yield

$$\frac{\partial \tau_{ik}}{\partial x_k} + \mu_e [\operatorname{curl} \boldsymbol{H} \times \boldsymbol{H}]_i = 0 , \tag{1.13}$$

$$\nabla^2 H_i = 0 , \tag{1.14}$$

$$k \nabla^2 T_0 + Q = 0 . \tag{1.15}$$

We consider the case of plane equilibrium with $\boldsymbol{u} = (u_1, u_2, 0)$, $\boldsymbol{H} = (H_1, H_2, 0)$ and $\tau_{12} = \tau_{21}$. The stress function χ_0 may be introduced consistent with (1.15) with

$$\tau_{12} = \frac{-\partial^2 \chi_0}{\partial x_1 \partial x_2} - \mu_e H_1 H_2 , \tag{1.16}$$

$$\tau_{11} = \frac{\partial^2 \chi_0}{\partial x_2^2} - \frac{1}{2} \mu_e [H_1^2 - H_2^2] , \tag{1.17}$$

$$\tau_{22} = \frac{\partial^2 \chi_0}{\partial x_1^2} + \frac{1}{2} \mu_e [H_1^2 - H_2^2] . \tag{1.18}$$

The compatibility relation between the plane strain components gives

$$\nabla^4 \chi_0 + \frac{1}{2(1-\nu)} \mu_e \left[\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) (H_1^2 - H_2^2) + 4 \frac{\partial^2}{\partial x_1 \partial x_2} (H_1 H_2) \right] + \\ + \frac{2\mu\alpha(1+\nu)}{(1-\nu)} \nabla^2 T_0 = 0 . \tag{1.19}$$

It may be shown that the governing equation (1.19) can be solved without any recourse to the usual linearization [**8**] viz: $H_1 = H_0 + h_x$, $H_2 = h_y$. In fact with the complex substitution $z = x + iy$, $\bar{z} = x - iy$, equation (1.14) gives

$$H_1 + iH_2 = iAz + \overline{h(z)}\,, \tag{1.20}$$

where $h(z)$ is an arbitrary function of z and A is a real constant, while eq. (1.19) provides

$$\nabla^4 \chi_0 + \alpha E_0 \nabla^2 T_0\,, \qquad E_0 = \frac{2\mu_0(3\lambda_0 + 2\mu_0)}{\lambda_0 + 2\mu_0} \left(\nabla^2 \equiv \frac{\partial^2}{\partial z \partial \bar{z}}\right). \tag{1.21}$$

The eq. (1.21) may be replaced by the simple biharmonic equation

$$\nabla^4(\chi_0 - \chi_1) = 0\,, \tag{1.22}$$

where χ_1 satisfies

$$\nabla^2 \chi_1 + \alpha E_0 T_0 = 0\,. \tag{1.23}$$

Hence from (1.22), we obtain

$$\chi_0 = \chi_1 + f_2(z) + \overline{f_2(z)} + \bar{z} f_3(z) + z \overline{f_3(z)}\,, \tag{1.24}$$

while eqs. (1.16)-(1.18) now provide the components of stress for the magneto-thermoelastic problem.

3 - Magneto-thermoviscoelasticity and the correspondence principle

The governing equations of magneto-thermoviscoelasticity differ from those of the magneto-thermoelasticity eqs. (1.1)-(1.12), in that the constitutive eq. (1.1) must now be replaced by the thermoviscoelastic relation [7]

$$\begin{aligned} P_1(D)P_3(D)\tau_{ij} &= P_2(D)P_3(D)e_{ij} \\ &+ \delta_{ij}[\tfrac{1}{3}\{P_1(D)P_4(D) - P_2(D)P_3(D)\}e - P_1(D)P_4(D)\alpha T] \qquad (i, j = 1, 2, 3)\,, \end{aligned} \tag{1.25}$$

where $P_i(D)$, $(i = 1, 2, 3, 4)$ are linear differential operators

$$P_i(D) = \sum_{n=0}^{N_i} a_i^{(n)} D^n\,, \qquad D^n \equiv \frac{d^n}{dt^n} \qquad (i = 1, 2, 3, 4)\,, \tag{1.26}$$

$a_i^{(N_i)}$ are certain constants. For a perfectly elastic body the operators $P_i(D)$ reduce to the first term of the series (1.26)

$$a_1^{(0)} = 1\,, \quad a_2^{(0)} = 2\mu\,, \quad a_3^{(0)} = 1\,, \quad a_4^{(0)} = \frac{2\mu(1+\nu)}{1-2\nu}\,. \tag{1.27}$$

The constitutive relations (1.25) may also be represented in integral form [7]

$$\tau_{ij} = 2\int_0^t a(t-\tau)\dot{e}_{ij}\,d\tau + \delta_{ij}\int_0^t [b(t-\tau)\dot{e} - \{3b(t-\tau) + 2a(t-\tau)\}\alpha\dot{T}]\,d\tau\,. \tag{1.28}$$

It is assumed that the viscoelastic body is free of stresses at the initial instant; $a(t)$ and $b(t)$ are some functions of time which for perfectly elastic body reduce to the Lamé constants μ and λ. The dot in the integrand denotes differentiation with respect to τ, while $e = e_{ii}$.

For identical magneto-thermoviscoelastic problem eqs. (1.2)-(1.12) together with the boundary and initial conditions remain unchanged. Taking the Laplace transform of eqs. (1.1), (1.25) or (1.28) we obtain

$$\bar{\tau}_{ij} = 2\mu_0\bar{e}_{ij} + (\lambda_0\bar{e} - \beta_0\bar{T})\delta_{ij} \quad \text{(elastic)} \tag{1.29}$$

and

$$\bar{\tau}_{ij} = 2\bar{\mu}(p)\bar{e}_{ij} + (\bar{\lambda}(p)\bar{e} + \bar{\beta}(p)\bar{T})\delta_{ij} \quad \text{(viscoelastic)}\,, \tag{1.30}$$

where

$$\bar{\mu}(p) = \frac{P_2(p)}{2P_1(p)}\,, \qquad \bar{\lambda}(p) = \frac{P_1(p)P_4(p) - P_2(p)P_3(p)}{3P_1(p)P_3(p)} \tag{1.31}$$

and $\bar{\beta}(p) = (3\bar{\lambda} + 2\bar{\mu})\alpha$ or $\bar{\mu}(p) = p\bar{a}(p)$, $\bar{\lambda}(p) = p\bar{b}(p)$, while

$$\bar{\beta}(p) = (3\bar{\lambda} + 2\bar{\mu})\alpha \tag{1.32}$$

depending on which of the two forms (1.25) or (1.28) are considered for Laplace transform.

Also the Laplace transform of the magneto-thermoelastic relation (1.21) provides

$$\nabla^4\bar{\chi}_0 + \alpha E_0\nabla^2\bar{T}_0 = 0\,. \tag{1.33}$$

Due to magneto-thermoelastic and the magneto-thermoviscoelastic analogy shown above, the stress function χ for corresponding plane strain problem for viscoelastic body may be derived from

$$\nabla^4 \bar{\chi} + \alpha \bar{E}_0(p) \nabla^2 \bar{T}_0 = 0 . \tag{1.34}$$

A comparison of eqs. (1.33) and (1.34) under similar boundary and initial conditions shows that $\bar{\chi} = p\bar{f}(p)\bar{\chi}^0$, $\bar{f}(p) = \bar{E}(p)/pE_0$, whence it follows that

$$\chi(x_1, x_2, t) = \int_0^t f(t-\tau) \frac{\partial}{\partial \tau'} \chi_0(x_1, x_2, \tau') \, d\tau' . \tag{1.35}$$

If the temperature field in viscoelastic body is stationary the differential equation for $\bar{\chi}$ assumes the form

$$\nabla^4 \bar{\chi} + \alpha \frac{\bar{E}_0(p)}{p} \nabla^2 T_0 = 0 . \tag{1.36}$$

A comparison of eqs. (1.36) and (1.21) yields the relation

$$\bar{\chi} = \bar{f}(p) \chi_0 . \tag{1.37}$$

Hence $\chi = f(t)\chi_0(x_1, x_2)$. In absence of external heat sources ($Q = 0$), eq. (1.15) implies $\nabla^2 \tau_0 = 0$. Therefore $\chi = \chi_0$, and eqs. (1.16), (1.18) now provide stress components for both the systems.

References

[1] T. Alfrey, *Non-homogeneous stresses in viscoelastic media*, Quart. Appl. Math. **2** (1944), 113-119.

[2] D. R. Bland, *The theory of linear viscoelasticity*, Pergamon Press, Oxford 1960.

[3] D. S. Chandrasekharaiah, *The propagation of magneto-thermoviscoelastic plane waves in a parallel union of the Kelvin and Maxwell bodies*, Proc. Camb. Phil. Soc. **70** (1970), 343-350.

[4] H. H. Hilton, *An extension of Alfrey's elastic-viscoelastic analogy to viscoelastic thermal stress problems*, Rep. No. TSVE-TR-2, Dep. Aero. Eng., University of Illinois 1953.

[5] E. H. Lee, *Stress analysis in viscoelastic bodies*, Quart. Appl. Math. **13** (1955), 183-190.

[6] V. P. Madan, *The propagation of plane waves in a magneto-viscoelastic medium*, Tensor **21** (1970), 204-206.

[7] W. NOWACKI, *Thermoelasticity*, Addison-Wesley Publishing Company Inc., London 1962.

[8] G. PARIA, *Magneto-elasticity and magneto thermal elasticity*, Advances in Applied Mechanics, Academic Press, New York **10** (1967), 73-110.

[9] R. V. SOUTHWELL, *An introduced to the theory of elasticity*, Oxford Univ. Press 1941.

[10] E. STERNBERG, *On transient thermal stresses in linear viscoelasticity*, Proc. Third U.S. Nat. Congr. Appl. Mech., New York 1958.

Abstract

It is shown the solution to certain steady plane strain problem in magneto-thermoviscoelasticity may be obtained with the aid of a correspondence principle, using the solution of the associated magnetothermoelastic problem. Linear differential as well as integral operator forms of the constitutive relations are used to represent the thermovisco-elastic behaviour; while the physical properties of the material such as thermal coefficient α *and the permeability* μ_e *etc. are assumed independent of time.*

* * *