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Convexity of the free boundary in some classical parabolic free boundary problems (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

Consider the problem

$$z_{xx} - z_t = 0 \quad \text{in} \quad D_T = \{(x, t): 0 < x < s(t), 0 < t < T\}, \tag{1.1}$$

$$s(0) = 1, \tag{1.2}$$

$$z(x, 0) = h(x), \quad 0 < x < 1, \tag{1.3}$$

$$z_x(0, t) = 0, \quad 0 < t < T, \tag{1.4}$$

$$z(s(t), t) = 0, \quad 0 < t < T, \tag{1.5}$$

$$\dot{s}(t) = - z_x(s(t), t), \quad 0 < t < T \tag{1.6}$$

in the case $h(x) \leqslant 0$. This problem is often referred to as a mathematical scheme for the freezing of a supercooled liquid (although this simple scheme for such a non-equilibrium phenomenon is far from being satisfactory). Problem (1.1)-(1.6) has been widely investigated and it is known to be well posed for suitable T, provided h satisfies some regularity conditions (see [**1**], [**2**], [**5**]).

In what follows we shall need some more results on (1.1)-(1.6). First we

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recall that $s(t)$ is analytic for $t > 0$ (see [3]) and strictly decreasing (except for the trivial case $h \equiv 0$, in which $z \equiv 0$, $s \equiv 1$). Then, z is infinitely differentiable up to the boundary. Differentiating (1.5) along $x = s(t)$ and using (1.1)-(1.6), we obtain

$$z_{xx}(s(t), t) = \dot{s}^2(t)\,, \qquad 0 < t < T\,. \tag{1.7}$$

Remark 1.1. It does not exist any interval (t_1, t_2), $0 \leqslant t_1 < t_2 \leqslant T$, where $\dot{s} \equiv K$. As a matter of fact, in this case $z(x, t)$ would coincide in (t_1, t_2) with the solution of the non-characteristic Cauchy problem for the heat equation with data $z(s(t), t) = 0$, $z_x(s(t), t) = -K$: this solution is known to exist (see [4]) for any x and to have non vanishing x-derivative. Thus the boundary condition $z_x(0, t) = 0$ would be contradicted.

In [2] A. Friedman and R. Jensen have studied (1.1)-(1.6) from the point of view of the convexity of the free boundary under some special assumption on $h(x)$. Their main tool was the introduction of the function

$$v = z_{xx}/z_x \tag{1.8}$$

and the study of its level curves.

In the present paper the investigation on the convexity of the free boundary of (1.1)-(1.6) is further developed and a much wider class of initial data is considered (§ 3).

The classical Stefan problem (with $h \geqslant 0$) and more general free boundary problems could also be considered with parallel arguments. On the contrary, nontrivial modifications are needed to handle the oxygen diffusion-consumption problem (see [1]), to which § 4 is devoted.

2. - Preliminary results

Assume

$$h(x) \leqslant 0\,, \qquad x \in [0, 1]\,; \qquad h(1) = 0\,; \qquad h \in C_2[0, 1] \tag{2.1}$$

and define

$$M = \{x : x \in [0, 1],\, h'(x) \neq 0\}\,, \tag{2.2}$$

which is non-void if we exclude the trivial case $h \equiv 0$. Set

$$H(x) = h''(x)/h'(x)\,, \qquad x \in M \tag{2.3}$$

and note that if (s, z) solve (1.1)-(1.6), then

$$v_{xx} + 2vv_x - v_t = 0 \quad \text{in } D_T, \tag{2.4}$$

$$v(s(t), t) = -\dot{s}(t), \qquad t \in (0, T), \tag{2.5}$$

$$v_x(s(t), t) = \ddot{s}(t)/\dot{s}(t), \qquad t \in (0, T), \tag{2.6}$$

$$v(x, 0) = H(x), \qquad x \in M. \tag{2.7}$$

This follows from definitions (1.8), (2.2), (2.3).

The main subject of this section will be the investigation of the behaviour of $v(x, t)$ near points $(x_0, 0)$ with $x_0 \in \partial M$, which will be called *singular points.*

(A) *Classification of the singular points.* We shall consider the following cases.

Case 1.1. $x_0 \in (0, 1)$, h'' changes its sign at x_0.
Case 1.2. $x_0 \in (0, 1)$, h'' does not change it sign across x_0.
Case 2.1. $x_0 = 0$, $h'(0) = 0$.
Case 2.2. $x_0 = 0$, $h'(0) \neq 0$.
Case 3.1. $x_0 = 1$, $h'(1) = 0$.
Case 3.2. $x_0 = 1$, $h'(1) > 0$.

In cases 1.1 and 1.2 we assumed $h'(x) \neq 0$ in a neighborhood of x_0 but the analysis can be extended to cover the cases in which $h' = 0$ in some half neighborhood of x_0.

Concerning the other cases, note that $x_0 = 0$, $x_0 = 1$ do not belong to ∂M if $h' = 0$ in a half neighborhood of 0 and 1. Our interest will be focused on the behaviour of the level curves of $v(x, t)$ corresponding to non-negative (positive) values of v, henceforth called « non-negative (positive) level curves » for sake of brevity; but similar investigation can be performed, on the negative level curves.

Note that whenever $H(x)$ is monotonic in a right neighborhood $(x_0, x_0 + \varepsilon)$ of an interior singular point (see cases 1.1 and 1.2), then

$$\lim_{x \to x_0 +} H(x) = +\infty. \tag{2.8}$$

Similarly, if $H(x)$ is monotonic in $(0, \varepsilon)$ then

$$\lim_{x \to 0_+} H(x) = +\infty. \tag{2.9}$$

The cases to which we shall apply the analysis of this section are such that the above conditions are satisfied. Consequently we shall assume (2.8)-(2.9) hold henceforth.

(B) *Study of case* 1.1. In this case h' does not change its sign at x_0 and hence the maximum principle applied to z_x shows that no level curve for z_x originates from $(x_0, 0)$. On the other hand, there is a curve γ in D_T originating from $(x_0, 0)$ where $z_{xx} = 0$. Because of (2.8) there exists a family of positive level curves for v originating from $(x_0, 0)$ and lying on the right of γ. Consider the level curve γ_k corresponding to the value $k > 0$: this is the zero level curve for the function $w_k(x, t) = z_{xx}(x, t) - kz_x(x, t)$, which is a solution of the heat equation, continuous in $|x - x_0| \leqslant \sigma$, $0 \leqslant t \leqslant \sigma_1$, for some $\sigma, \sigma_1 > 0$. Recalling that $h''(x)$ and $h'(x)$ have the same sign in a right half neighborhood of $x = x_0$, (2.8) implies that there exists $\sigma > 0$ such that for $x - x_0 \in (0, \sigma)$

(a) if $h'(x) > 0$, then $w_k(x, 0) \geqslant 0$ and $w_k(\sigma, 0) > \varepsilon$ for some $\varepsilon > 0$;
(b) if $h'(x) < 0$, then $w_k(x, 0) \leqslant 0$ and $w_k(\sigma, 0) < -\varepsilon$.

By continuity a σ_1, can be determined such that for $t \in (0, \sigma_1)$, $w_k(\sigma, t) < -\varepsilon/2$ or $w_k(\sigma, t) > \varepsilon/2$ in the respective cases (b), (a); the maximum principle implies that in the region $D(\gamma_k, \sigma, \sigma_1)$ bounded by γ_k, $x = x_0 + \sigma$, $t = 0$, $t = \sigma_1$, $w_k(x, t)$ has the same sign as $h'(x)$ for $x - x_0 \in (0, \sigma)$. This proves that irrespectively of the sign of $h'(x)$ in a neighborhood of $x = x_0$ it is $v(x, t) > k$ in $D(\gamma_k, \sigma, \sigma_1)$, i, e, that no level curves for v carrying values less then k can originate from $(x_0, 0)$ on the right of γ_k. Consequently, *the positive level curves of $v(x, t)$ originating from $(x_0, 0)$ are ordered, with values increasing from γ to the right.*

(C) *Study of case* 1.2. We claim that no level curve of v originates from $(x_0, 0)$. Indeed, in this case there exists a curve $\tilde{\gamma}$ starting from $(x_0, 0)$ on which $z_x = 0$. For any $k > 0$ consider the function $w_k(x, t)$ introduced above. Since in a neighborhood of $(x_0, 0)$ z_{xx} has the same sign as $h''(x)$ and z_x has this same sign on the right of $\tilde{\gamma}$, applying the maximum principle we find that for suitable σ, $\sigma_1 > 0$ it is $v(x, t) > k$ in the region $D(\tilde{\gamma}, \sigma, \sigma_1)$. This means that $\lim_{(x,t)\to(0,0)} v(x, t) = +\infty$ along any path lying on the right of $\tilde{\gamma}$.

(D) *Study of case* 2.1. We are in the case just considered, after a reflection about the t-axis.

(E) *Study of case* 2.2. Assume $h'(0) > 0$ to be specific and write

$$z_x(x, t) = U(x, t) + w(x, t),$$

where U solves the heat equation in the half-plane $t > 0$ with initial datum

$$U(x, t) = \begin{cases} -h'(0) & \text{if } x < 0 \\ h'(0) & \text{if } x > 0 , \end{cases}$$

and w results to be a smooth solution of $w_{xx} - w_t = 0$ (in the domain D_T reflected about the t-axis) such that

$$\lim_{(x,t)\to(0,0)} w(x, t) = 0 , \qquad \lim_{(x,t)\to(0,0)} w_x(x, t) = h''(0) .$$

It is immediately seen that $U(x, t) = h'(0) \operatorname{erf} x/2\sqrt{t}$. Therefore, if lim inf $U = 0$ as (x, t) approaches $(0, 0)$ along a curve λ_1, the same will be true for any curve λ_2 lying on the left hand side of λ_1. Moreover $U_x(x,t) = (\pi t)^{-1/2} h'(0) \exp(-x^2/4t)$, and hence if $\lim U_x = +\infty$ along a curve $\bar{\lambda}_1$ originating from $(0, 0)$ the same will be true for any curve $\bar{\lambda}_2$ lying on its left hand side. Now, assume $\limsup v = +\infty$ along a curve λ. Then $\liminf z_x = 0$ and/or $\limsup z_{xx} = +\infty$ along λ. This implies $\liminf U = 0$ and/or $\limsup U_x = +\infty$ along λ. But this means that $\limsup v = +\infty$ along any curve $\bar{\lambda}$ on the left of λ. Conversely, on the right hand of a level curve for v originating from $(0, 0)$ it cannot exist a path along which v is unbounded.

Thus, we can exclude the possibility of «loops» for the level curves of v and (because of the maximum principle) we can affirm that *the level curves of v originating from $(0, 0)$ are ordered and their values increase towards the left from the value $H(0)$ to $+\infty$.*

(F) *Study of case* 3.1. Recalling (2.1) and (1.7), note that z_{xx} assumes bounded values of both signs in a neighborhood of $(1, 0)$. In the region bounded by the free boundary and the level curve $z_{xx} = 0$ originating from $(1, 0)$ it is $0 < z_{xx} < \dot{s}^2$. Denote by $\delta(t)$ the horizontal width (for «small» values of t) of this region and observe that $z_x > -\dot{s} - \dot{s}^2\delta(t)$. Since $\dot{s}(0) = 0$ this shows that v is bounded and continuous in this region and that *no positive level curve for v originates from* $(1, 0)$, because of the maximum principle.

(G) *Study of case* 3.2. Because of (1.7) we have that if $h'(1) = H(1)$, z_{xx} — and hence v — are continuous at $(1, 0)$. If $H(1) > h'(1)$, *then the level curves corresponding to the values between $h'(1)$ and $H(1)$ originate from* $(1, 0)$ *and are ordered decreasing rightwards*, because of the maximum principle.

For the same reason, when $H(1) < h'(1)$ they are increasing rightwards.

Remark 2.1. In case 3.2, if $H(1) < h'(1)$ there exists a $\tau > 0$ such that

$$\ddot{s}(t) < 0\,, \qquad t \in (0, \tau)\,. \tag{2.11}$$

Indeed, for sufficiently small t, $v_x(s(t), t)$ is positive and (2.11) is proved because of (2.6).

Note that (2.11) implies that $v(s(t), t)$ increases with t in $(0, \tau)$ and hence no level curve for v originating from points $(s(t), t)$ with t in $(0, \tau)$ can leave D_T at the point $(1, 0)$.

By a similar argument we prove that if $H(1) > h'(1)$ in case 3.2, then there exist a $\tau > 0$ such that

$$\ddot{s}(t) > 0\,, \qquad t \in (0, \tau) \tag{2.12}$$

and no level curve for v can join (1.0) with $(s(t), t)$ with t in $(0, \tau)$.

3. - The curvature of the free boundary

First, we need

Lemma 3.1. *Let* (s, z) *solve* (1.1)-(1.6) *and let* v *be defined according to* (1.7). *No local estremum of* v *can be attained on* $x = s(t)$ for $t \in (0, T)$.

Proof: see [**2**]. Next, consider a level curve γ of v originating from $x = s(t)$, $t > 0$. It is clearly a positive level curve (see (2.5)); moreover if $\ddot{s}(t) \neq 0$ the derivatives of v along the x direction and along the free boundary have the same sign (see (2.6)). This means that, starting from the free boundary, the t coordinate along γ is increasing for a finite interval. Hence, only one level curve for v can originate from a point of the free boundary (recall the maximum principle) unless $\ddot{s}(t) = 0$. Moreover, t cannot have a minimum on γ in D_T, (because of the maximum principle) nor can be $t = \text{const.}$ between two points of γ (since it would contradict the local analiticity of v with respect to x). Then, only the two following cases can occur: (a) starting from the free boundary, t is increasing along γ; (b) t is first increasing, then decreasing along γ. We have the following

Lemma 3.2. *Let* γ *be a level curve for* v *originating from the free boundary at* $t = t^*$. *If the case* (a) *occurs, the free boundary cannot have inflection points for* $t > t^*$.

Proof. Consider the domain bounded by γ, the free boundary and the line $t = \tilde{t} > t^*$. Since $v \not\equiv \dot{s}(t^*)$, a minimum or a maximum must be attained; because of Lemma 3.1 it can be attained only at $(s(\tilde{t}), \tilde{t})$. Since $\tilde{t}$ is arbitrary, the lemma is proved. The analysis of § **2** together with the above lemmas allows us to investigate the curvature of the free boundary.

To be more specific, we shall begin by studying some particular cases. First, we note that if $H(x) \geqslant 0$ in M, then $h'(x)$ is non negative and nondecreasing in M and in particular $h'(1) > 0$, unless $h \equiv 0$. We have

Theorem 3.3. *Assume* (2.1) *is satisfied and* $H(x)$ *is nonnegative and nonincreasing in* M. *Then*

(i) *the free boundary can have at most one inflection point;*

(ii) *if* $H(1) > h'(1)$, *then* $\ddot{s}(t) > 0$ *for any* $t \in (0, T)$, *and* (1.1)-(1.6) *is solvable for arbitrary* $T > 0$;

(iii) *if* $H(1) = h'(1)$, *then* (1.1)-(1.6) *is solvable for arbitrary* $T > 0$ *and there exists a* $t_0 \geqslant 0$ *such that* $\ddot{s}(t) \leqslant 0$ *for* $t \leqslant t_0$ *and* $\ddot{s}(t) > 0$ *for* $t > t_0$;

(iii)′ *if in case* (iii) *there exists* $H'(1)$, *then* t_0 *can be positive only if* $H'(1)=0$;

(iv) *if* $H(1) < h'(1)$, *then the free boundary has an inflection point as in case* (iii) *if and only if* $\int_0^1 h(x)\,\mathrm{d}x > -1$.

Proof of (i). Under the foregoing assumptions $M = (\theta, 1]$ with $\theta \in [0, 1)$ (we exclude the trivial case $h \equiv 0$). Consider the level curves for v originating from the free boundary: as long as the case (b) occurs, they exit D at points $(x, 0)$ with $x < 1$ (recall also Remark 2.1). Thus $\dot{s}(t)$ decreases for increasing t because of the assumptions on $H(x)$ and the results of § **2** (cases 1.2-2.1).

When case (a) occurs, then Lemma 3.2 applies. Hence there can exist only one inflection point separating the two «families» of level curves.

Proof of (ii). Setting $h'(1) = \alpha > 0$, we have

$$h'(x) = \alpha \exp\Big(-\int_x^1 H(y)\,\mathrm{d}y\Big) > 0 \qquad x \in (\theta, 1],$$

and

$$h(x) = -\alpha \int_x^1 \exp\Big(-\int_\mu^1 H(y)\,\mathrm{d}y\Big)\,\mathrm{d}\mu \qquad x \in (\theta, 1],$$

since $h(1) = 0$. But we assumed $H(x) \geqslant H(1) > h'(1) = \alpha$, whence

$$h(x) > -1 + \exp\big(-\alpha(1 - x)\big) \qquad x \in [0, 1], \tag{3.1}$$

(if $\theta > 0$, $h'(x) \equiv 0$ in $(0, \theta)$). Therefore, $\int_0^1 h(x)\,dx > -1$ and from theorem 2.9 of [**1**] it follows that (1.1)-(1.6) is solvable for any positive T (recall $h' \geqslant 0$). Thus, $\ddot{s}(t) > 0$ for sufficiently large t and (i) together with (2.12) concludes the proof of (ii).

Proof of (iii). The proof is almost the same. The inequality (3.1) holds with $\geqslant$ in place of $>$. Therefore (1.1)-(1.6) is solvable for any $T > 0$, $\ddot{s}(t) > 0$ for large t and the conclusion follows from (i).

Proof of (iii)$'$. If $H'(1)$ exists then $H'(1) \leqslant 0$ because of the assumption on $H(x)$ and $\ddot{s}(0) = -h'(1)H'(1)$ from (2.6). Hence $H'(1) < 0$ implies $\ddot{s}(0) > 0$ and we are in the same conditions as in (ii).

Proof of (iv). Recall (2.11) and (i). If $\int_0^1 h(x)\,dx > -1$, then (1.1)-(1.6) is solvable for any T (theorem 2.9 of [**1**]) and $x = s(t)$ must have an inflection point. Conversely, assume the free boundary has an inflection point (and only one because of (i)) for $t = t_0 > 0$, say. Because of (2.11), this means that $\ddot{s}(t) > 0$ for $t > t_0$. Therefore $\dot{s}$ is bounded from below for $t > t_0$: this is enough to prove that the solution of (1.1)-(1.6) exists for any T (recall theorems 2.9 and 2.12 of [**1**]).

Remark 3.4. Cases considered in theorem 2.1 of [**2**] are included in (iii) (for the particular case $h'(0) = 0$) and more precise informations on such cases are found in (iii)$'$. It is also worth to note that the assumptions on h imply the global existence of the solution of problem (1.1)-(1.6): this is of some relevance in connection with assumptions (2.8) and (2.25) of [**2**].

The next step will be to generalize the assumptions on $H(x)$ allowing it to become negative, still confining ourselves, for the time being, to cases in which $h'(x) \geqslant 0$. Let $P = \{x \in M : H(x) \geqslant 0\}$. We have the following

Theorem 3.5. *Assume* (2.1) *is satisfied and* $H(x)$ *is nonincreasing in* P. *Then*

(i) *if* $P = \emptyset$ *or* $P = [0, b]$, $0 \leqslant b \leqslant 1$, *the free boundary can have at most one inflection point*;

(ii) *if* $P = \bigcup_{k=1}^{n} (a_k, b_k)$, $(0 \leqslant a_k < b_k \leqslant 1,\ b_k < a_l$ *for* $k < l)$ *and if* $h'(x) \geqslant 0$ *in* $[0, 1]$, *the free boundary can have at most* $2n + 1$ *inflection points. This number reduces to* $2n - 1$ *if* $a_1 = 0$ *or if* $h'(x) = 0$ *in* $[0, a_1]$.

Proof of (i). In this case ∂M contains only the point $x = 0$ and a point $x = b^* \in (b, 1]$. Recalling the results of **2**, we have that the positive level curves of v originating from $t = 0$, if any, are ordered and correspond to values decreasing toward the right. Hence the argument of Theorem 3.3 (i) applies.

Proof of (ii). Now we can have n singular points in $(0, 1)$ ($n-1$, if $a_1=0$) which are as in case 1.1, since $h'(x)\geqslant 0$. Let $(0, \tilde{t})$, $\tilde{t}\geqslant 0$, be the time interval such that the level curves for v originating from the free boundary have a behaviour of type (b) (i.e. they exit D at $t = 0$). As long as their exit points $(x, 0)$ are non singular, the corresponding values of v increase as t increases and then $\ddot{s}(t) < 0$. But, if the level curves originating from $(s(t), t)$ for t in a certain interval (t_1, t_2) leave D at a singular point, then the corresponding values decrease as t increases (as we say in § **2**). Then, if this singular point is not a_1, it has at its left hand side another segment where $H(x)$ is positive and nonincreasing, so that it can be $\ddot{s}(t) < 0$ for $t > t_2$. Therefore, each a_i ($i = 2, ..., n$) can generate two inflection points for the free boundary. The same situation arises in a_1, if $a_1 \neq 0$ and $h'(x) \neq 0$ in $[0, a_1]$ (see the analysis of § **2** concerning the point $(0, 0)$). If $a_1 = 0$ or if $h'(x) = 0$ in $[0, a_1]$, it does not correspond to any inflection point. Thus we can have in $(0, \tilde{t})$ $2n$ inflection points (or $2n - 2$ if $a_1 = 0$). Finally, another inflection point is the one from which the level curve separating families (a) and (b) can originate.

The results of Theorem 3.5 easily generalize to the case in which, retaining the assumption $h'(x)\geqslant 0$, one allows the sign of the slope of $H(x)$ to change in P. In this case points a_k are not necessarily singular: let m be the number of points a_k such that $h'(a_k) = 0$. We have

Theorem 3.6. *Assume* (2.1) *is satisfied. If* $h'(x)\geqslant 0$ *in* $[0, 1]$ *and if the sign of the slope of* $H(x)$ *changes* k *times in* P, *then the free boundary can have at most* $2m + k + 1$ *inflection points, which reduces to* $2m + k - 1$ *if* $H(x)$ *is positive and nondecreasing in a neighborhood of the origin.*

Proof. The proof is essentially the same as in the previous theorem.

Remark 3.7. If $H(1) > h'(1)$, we saw that $\ddot{s}(t) > 0$ in an interval $(0, \tau)$: this means that the corresponding level curves either are of the family (a) from the very beginning or that they leave D at a singular point a_j. In any case the number of admissible inflection points is one less.

To complete our investigation, we have now to allow $h'(x)$ to change its sign, i.e. to include singular points of the type considered in case 1.2. Here, the same procedure is applicable, but the situation is much more complicated so that is seems more reasonable not to state a general theorem, and to confine ourselves to the main changes which can occur. In what follows we shall assume that near singular points h'' is such that (2.8), (2.9) are valid.

Let x_0 be a singular point as in case 1.2 and consider the level curve Γ, where $z_x = 0$, originating from it. We can have three possibilities

(i) Γ leaves D_T at a point $(0, t_0)$;

(ii) Γ leaves D_T at a point $(x_1, 0)$, which is necessarily a singular point as in case 1.2;

(iii) Γ remains in D_T till $t = T$.

In this third case, which is the simplest one, only the behavior of $h(x)$ for $x > x_0$ determines the curvature of the free boundary. If the second case occurs, assuming $x_1 < x_0$, there is at least one point $\bar{x}$, $x_1 < \bar{x} < x_2$, such that $h''(\bar{x}) = 0$ and h'' changes its sign across $\bar{x}$. Consider the level curve $z_{xx} = 0$ originating from $(\bar{x}, 0)$ and intersecting Γ at $(\hat{x}, \hat{t})$ (we shall have one of such points at least at the top of Γ). At this point the analysis performed for the singular points in case 1.1 applies, and one finds that the behavior of $h(x)$ in the interval (x_0, x_1) does not affect the curvature of the free boundary. Finally, in the first case the curve Γ « prevents » the level curves of v originating from the interval $(0, x_0)$ of the x-axis from hitting the free boundary and thus from influencing its curvature.

4. - The case of the oxygen-consumption problem

As it has been pointed out in [1] the oxygen-consumption problem corresponds to a problem of type (1.1)-(1.7) for the t derivative $z = u_t$ of the oxygen concentration, but the initial datum for z behaves like a « δ-function », if the initial concentration of the oxygen corresponds to the stationary level $u(x, 0) = \frac{1}{2}(x-1)^2$. Hence, the analysis just performed does not apply directly. Consider representation (3.38) of [1]; differentiating both sides w.r.t. x, one obtains the following expression for the time derivative of the oxygen concentration

$$z(x, t) = -N(x, t; 0, 0) - \int_0^t \dot{s}(\tau) N(x, t; s(\tau), \tau)\, d\tau \,. \tag{4.1}$$

Confining our attention to a neighborhood of the origin, we find that the asymptotic behaviour of the successive derivatives of z as $(x, t) \to (0, 0)$ is

$$\frac{\partial^n z}{\partial x^n} \sim -\frac{\partial^n}{\partial x^n} N(x, t; 0, 0) \,. \tag{4.2}$$

Thus

$$v(x,t) \sim \frac{N_{xx}(x,t;\,0,0)}{N_x(x,t;\,0,0)} = \frac{1}{x} - \frac{x}{2t}$$

Therefore, the curve $v = 0$ behaves near $(0, 0)$ like the parabola $x^2 = 2t$ and on its left the level curves for v originating from $(0, 0)$ are ordered and correspond to values increasing leftwards ($v_x \sim -(1/x^2) - (1/2t)$ is negative on the left of $x^2 = 2t$). On the other hand, there are no positive level curves originating from $(1, 0)$: this can be seen by the same argument we used in handling case 3.1 of § **2**.

Now, consider the level curves for v originating from the free boundary. As long as they are of type (b) (see § **3**) they exit D_T at $(0, 0)$ and hence $\ddot{s}(t) < 0$. If the curves originating from $(s(t), t)$ are of type (a) for $t \geqslant \tilde{t} \geqslant 0$, Lemma 3.1 ensures that no inflection points can exist for $t > \tilde{t}$ and therefore $\ddot{s}(t) < 0$, since $\lim_{t \to T^-} \dot{s}(t) = -\infty$.

This complete the proof of the following

Theorem 4.1. *The free boundary of the oxygen-consumption problem with stationary initial concentration is concave.*

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