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A. FASANO and M. PRIMICERIO (*)

Cauchy type free boundary problems for nonlinear parabolic equations (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

1.1. - The Cauchy type free boundary problem (problem (C))

It is well known that in many branches of applied mathematics (such as optimal stopping time problems, biomechanics, fluid flow in porous media, etc.) the following class of parabolic free boundary problems is encountered.

Problem (C). *Find a triple* (T, s, u) *such that*

(i) $T > 0$, $s(t)$ *is continuous and positive for any* t *in* $[0, T]$,

(ii) $u(x, t)$ *is continuous in the closure of the domain* $D_T = \{(x, t): 0 < x < s(t),\ 0 < t < T\}$, $u_x(x, t)$ *is continuous for* $0 < x \leqslant s(t)$, $0 < t < T$, u_{xx}, u_t *are continuous in* D_T,

(iii) *the following equations are satisfied*

$$a(x, t, u, u_x, s)\, u_{xx} - u_t = q(x, t, u, u_x, s) \quad in \quad D_T, \tag{1.1}$$

(*) Indirizzo: Istituto Matematico « U. Dini », Università, Viale Morgagni 67a, 50134 Firenze, Italy.

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(1.2) $s(0) = b > 0\,,$ (1.3) $u(x, 0) = h(x)\,, \quad 0 < x < b\,,$

(1.4) $u(0, t) = \varphi(t)\,,$ (1.5) $u(s(t), t) = f(s(t), t)\,, \quad 0 < t < T\,,$

(1.6) $u_x(s(t), t) = g(s(t), t)\,, \quad 0 < t < T\,,$

where $h(x)$, $\varphi(t)$, $f(x, t)$, $g(x, t)$ are prescribed functions for $x \geqslant 0$, $t \geqslant 0$ and the coefficients a, q are given functions of their arguments ($a > 0$).

Remark 1.1. The condition (1.4) on $x = 0$ can be replaced by

$$u_x(0, t) = \psi(t)\,, \tag{1.4)'}$$

with the substitution of (ii) with

(ii)' *$u(x, t)$ satisfies* (ii) *and u_x is continuous also at $x = 0$.*

Problem (C) in which the Cauchy data for u are assigned on $x = s(t)$, differs substantially from Stefan-like problems where the value of u is prescribed on the free boundary $x = s(t)$ and an explicit relationship between $\dot{s}(t)$ and $u_x(s(t), t)$ is given. Nevertheless, if the parabolic equation (1.1) reduces to

$$u_{xx} - u_t = q(x, t)\,,$$

Problem (C) can be transformed into a free boundary problem of the type studied in [2]$_1$, provided that one of the following conditions is fulfilled

$$f_x(x, t) - g(x, t) \neq 0 \qquad (x \geqslant 0,\ t \geqslant 0)\,, \tag{1.7}$$

$$f_x(x, t) - g(x, t) = 0\,, \quad f_{xx}(x, t) - f_t(x, t) \neq q(x, t) \qquad (x \geqslant 0,\ t \geqslant 0)\,, \tag{1.7)'}$$

(all the terms appearing in (1.7), (1.7)' are supposed to be continuous).

Contradicting the above assumptions may yield non uniqueness or nonexistence or noncontinuous dependence of the solution upon the data (see [**1**] and [**5**]).

We shall see that for the nonlinear case a similar analysis can be performed. However the free boundary problems we shall be lead to consider differ from the type studied in [2]$_1$ in the fact that higher order derivatives (e.g. u_{xx}, u_{xxx}) appear in the free boundary conditions: such problem will be referred to as *Stefan type problems of order* 2, 3, etc.

1.2. - Classes of regular solutions to problem (C)

We define the following classes of solutions of Problem (C) possessing higher regularity.

Definition 1.2. *A solution* (T, s, u) *is said to belong to the class* $\mathscr{S}_1$ *if, besides* (i)-(iii),

(iv) $s(t)$ *is continuously differentiable in* $(0, T)$,

(v) u_{xx}, u_t *are continuous up to* $x = s(t)$, $t > 0$.

Definition 1.3. *A solution* (T, s, u) *is said to belong to the class* $\mathscr{S}_2$ *if, besides* (i)-(v),

(vi) u_x *is continuous up to* $x = s(t)$, $t \geqslant 0$,

(vii) u_{xxx}, u_{xt} *are continuous for* $0 < x \leqslant s(t)$, $0 < t < T$.

Remark 1.4. In dealing with Problem (C) we always assume that

(A) h, φ, a *and* q *are continuous functions of their arguments,* $f(x, t)$ *is continuous for* $x > 0$, $t \geqslant 0$, *and* $g(x, t)$ *is continuous for* $x > 0$, $t > 0$. *Moreover the following conditions are to be satisfied*

$$h(0) = \varphi(0), \quad h(b) = f(b, 0) \tag{1.8}$$

(only the second condition is required when (1.4)′ is considered).

In dealing with solutions belonging to $\mathscr{S}_1$ we also require that

(A1) *the function* f *is continuously differentiable for* $t > 0$.

Finally, when solutions in $\mathscr{S}_2$ are considered, to (A), (A1) we add the requirement

(A2) g *is continuous for* $t \geqslant 0$ *and*

$$h'(b) = g(b, 0) ; \tag{1.9}$$

moreover g *has to be continuously differentiable for* $t > 0$ (1).

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(1) Conditions (A), (A1), (A2) are suggested by the definitions of Problems (C), (C1), (C2) and by Definitions 1.2, 1.3.

1.3. - First case of equivalence of problem (C) with a Stefan type problem of higher order

Assume that (T, s, u) is a solution of problem (C) belonging to $\mathscr{S}_1$ and suppose (A), (A1) are satisfied.

From (1.5), (1.6) we get

$$[(f_x - g)\dot{s} + f_t - u_t]_{x=s(t)} = 0 \,. \tag{1.10}$$

If (1.7) is satisfied, we can write (1.8) in the form

$$\dot{s} = A_1[s(t), t, u(s(t), t), u_x(s(t), t), u_{xx}(s(t), t)] \qquad (0 < t < T), \tag{1.11}$$

with

$$A_1 = [f_x(s(t), t) - g(s(t), t)]^{-1} \{a[s(t), t, u(s(t), t), u_x(s(t), t), s(t)] \cdot u_{xx}(s(t), t) - g[s(t), t, u(s(t), t), u_x(s(t), t), s(t)] - f_t(s(t), t)\}, \tag{1.12}$$

where $u(s(t), t)$ and $u_x(s(t), t)$ can be replaced by $f(s(t), t)$ and $g(s(t), t)$, respectively.

Now we state the following

Problem (C1). *Find a triple* (T, s, u) *satisfying* (i), (ii), (iv), (v) *and equations* (1.1), (1.2), (1.3), (1.4) [*or* (1.4)′ (2)], (1.6) *and*

$$\dot{s}(t) = A_1[s(t), t, u(s(t), t), g(s(t), t), u_{xx}(s(t), t)] \qquad (0 < t < T)\,. \tag{1.11)'$$

We have the following

Proposition 1.5. *If conditions* (A), (A1) *and* (1.7) *are satisfied, any solution* (T, s, u) *of Problem* (C1) *solves Problem* (C) *and belongs to* $\mathscr{S}_1$.

Proof. It suffices to show that (1.5) is satisfied. To this purpose note that

$$\frac{\mathrm{d}}{\mathrm{d}t} u(s(t), t) = [g\dot{s} + u_t]_{x=s(t)}\,,$$

owing to (1.6), whence

$$\frac{\mathrm{d}}{\mathrm{d}t} u(s(t), t) = \frac{\mathrm{d}}{\mathrm{d}t} f(s(t), t)\,,$$

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(2) In that case condition (ii) is modified according to Remark 1.1.

because (1.11)′ and (1.12) imply (1.10). Recalling the assumptions on f, the proof is completed.

Since the converse of Proposition 1.5 has been already proved, we have that looking for solutions of (C) in $\mathscr{S}_1$ is equivalent, under the above assumptions, to look for solutions to (C1).

1.4. - Second case of equivalence of problem (C) with a Stefan type problem of higher order

Now let us assume that $(T, s, u) \in \mathscr{S}_2$ solves Problem (C) and that (A), (A1), (A2), are satisfied.

We want to study the case in which instead of (1.7) we have

$$f_x(x, t) = g(x, t) . \tag{1.13}$$

This condition implies the differentiability of f_x; we assume that

$$a(x, t, f(x, t), p, s)\ f_{xx}(x, t) - f_t(x, t) \neq q(x, t, f(x, t), p, s) , \tag{1.14}$$

for any $x > 0$, $t \geqslant 0$, $s > 0$, $p \in (-\infty, +\infty)$.

Note that

$$u_t(s(t), t) = f_t(s(t), t) \qquad (0 < t < T), \tag{1.15}$$

$$[(af_{xx} - f_t - q)\dot{s} - au_{xt} + af_{xt}]_{x=s(t)} = 0 \qquad (0 < t < T) . \tag{1.16}$$

Thus

$$\dot{s}(t) = \Lambda_2[s(t), t, u(s(t), t), u_x(s(t), t), u_{xt}(s(t), t)] , \tag{1.17}$$

where

$$\begin{aligned} \Lambda_2 = \{&a[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]\, f_{xx}(s(t), t) \\ &- f_t(s(t), t) - q[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]\}^{-1} \\ &a[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]\{u_{xt}(s(t), t) - f_{xt}(s(t), t)\} , \end{aligned} \tag{1.18}$$

where $u(s(t), t)$ and $u_x(s(t), t)$ can be replaced by $f(s(t), t)$ and $g(s(t), t)$ respectively.

We state the following

Problem (C2). *Find a triple* (T, s, u) *satisfying* (i), (ii), (iv), (v), (vi), (vii) *and equations* (1.1)-(1.5) [*or* (1.4) (3)] *and*

$$\dot{s}(t) = \Lambda_2[s(t), t, f(s(t), t), u_x(s(t), t), u_{xt}(s(t), t)] . \tag{1.17$'$}$$

We have the following

Proposition 1.6. *If* (A), (A1), (A2), (1.13) *and* (1.14) *are satisfied, any solution of* (C2) *solves* (C) *in the class* $\mathscr{S}_2$.

Proof. It suffices to prove (1.6), i.e.

$$u_x(s(t), t) = f_x(s(t), t) . \tag{1.19}$$

It is easy to show that

$$\frac{d}{dt} u_x(s(t), t) = [a^{-1}(u_t - f_t)]_{x=s(t)}\, \dot{s}(t) + \frac{d}{dt} f_x(s(t), t) ,$$

where $[a]_{x=s(t)}$ must be considered as a known positive function of t.

On the other hand, differentiating (1.5) we get

$$(u_x - f_x)_{x=s(t)}\dot{s}(t) + (u_t - f_t)_{x=s(t)} = 0 .$$

Therefore the difference $X = (u_x - f_x)_{x=s(t)}$ satisfies the following linear differential equation

$$\dot{X}(t) = -[a^{-1}]_{x=s(t)}\, \dot{s}^2 X(t) \qquad (0 < t < T) ,$$

with zero initial value. This implies (1.19).

Since the converse of Proposition 1.6 has been already proved, the above stated assumptions guarantee that to find a solution of Problem (C) belonging to $\mathscr{S}_2$ is equivalent to solve (C2).

It is immediately seen that (C2) is equivalent to

Problem (C2)$'$. *Same as Problem* (C2) *with* (1.5) *replaced by* (1.15).

(3) See Remark 1.1.

1.5. - Further remarks and principal results

To conclude this section, let us consider the case

$$f_x(x,t) = g(x,t)\,, \qquad a(x,t,u,p,s)f_{xx}(x,t) - f_t(x,t) = q(x,t,u,p,s), \tag{1.20}$$

for any $x > 0$, $t \geq 0$ and any u, p $(-\infty, +\infty)$, $s > 0$.

Then in addition to (1.15) and (1.19), from (1.16) we have $u_{xt} = f_{xt}$ on the free boundary for any solution of Problem (C) such that u_{xt} is continuous up to the free boundary.

If we assume that f is infinitely differentiable and that a, q, u are also infinitely differentiable in a neighborhood of the free boundary, it is easy to show that the derivatives of any order of u at the free boundary coincide with the corresponding derivatives of f. This means that when (1.20) is satisfied we cannot find any relationship between $\dot{s}$ and some derivative of u.

On the other hand, as it has been pointed out for the linear case, Problem (C) is not well posed under (1.20). Hence we are clearly motivated to restrict our analysis to Problems (C1) and (C2).

In § **2** of the present paper we shall prove the well-posedness of Problem (C1), confining ourselves to the boundary condition (1.4) for the sake of brevity and setting

$$u_x(s(t),t) = 0\,, \tag{1.6)'$$

with obvious redefinition of the differential operator and of the data. For the sake of generality, we study the free boundary condition

$$\dot{s}(t) = \lambda_1[(s(t), t, u(s(t),t), u_{xx}(s(t),t))]\,, \tag{1.21}$$

with λ_1 prescribed independently of the other data and coefficients.

In § **3** we shall prove the well-posedness of problem (C2)′, where we set

$$u_t(s(t),t) = 0\,, \tag{1.15}$$

also in this case redefining the data and the coefficients. We shall study the more general free boundary condition

$$\dot{s} = \lambda_2[s(t), t, u(s(t),t), u_x(s(t),t), u_{xxx}(s(t),t)]. \tag{1.22}$$

2. - Problem (C1)

A simplified version of Problem (C1), namely with the coefficient a depending on u only, with no source term in the differential equation and with u_x entering the free boundary condition in a linear way, has been studied in [**2**]$_2$ with reference to a problem of liquid flow in porous media. In [**2**]$_2$ the boundary condition at $x = 0$ was assumed of the type (1.4).

In this section a different approach will be used to deal with the more general scheme introduced in § **1**.

For the sake of conciseness only the boundary condition (1.4) will be considered.

2.1. - Assumptions and notation of spaces and norms

For the notation of spaces and norms we refer to [**2**]$_3$, Sec. 2.

Here we add the spaces $C_{j,k}(\mathscr{D})$ $(j = 1, 2, 3; k = 0, 1)$—where $\mathscr{D}$ is a domain of $\boldsymbol{R}^2$—whose elements are the functions $u(x, t)$ having bounded continuous partial derivatives in $\mathscr{D}$ up to the j-th order w.r.t. x and to the k-th order w.r.t. t, with the norms $\|\cdot\|_{C_{j,k}(\mathscr{D})}$ defined as usual.

Concerning the data and the coefficient in Problem (C1), we shall assume

(α) $h \in H_{2+\alpha}[0, b]$, $\psi \in H_{1+\alpha}[0, \tilde{T}]$, $h'(0) = \psi(0)$, $h'(b) = 0$ for some given $\tilde{T} > 0$, $\alpha \in (0, 1)$;

(β) $a(x, t, u, p, s)$ is twice continuously differentiable (although w.r.t. t it is enough to require the Hölder continuity of a and of its derivatives w.r.t. the other arguments); for some continuous function $\mu(\xi, \eta)$, nondecreasing w.r.t. $\xi, \eta > 0$,

$$0 < \mu^{-1}(|u|, |p|) \leqslant a(x, t, u, p, s) \leqslant \mu(|u|, |p|),$$

$$\forall x \geqslant 0,\ u, p \in (-\infty, +\infty),\ t \in (0, \tilde{T}),\ s \geqslant 0;$$

(γ) $q(x, t, u, p, s)$ has the same differentiability properties as a;

(δ) $|\lambda_1(x, t, u, p)| \leqslant \nu(|u|, |p|)$, where ν is as μ and λ_1 is continuously differentiable.

2.2. - An auxiliary free-boundary problem

Set $R_{\tilde{T}}^{(b)} = [0, 3b/2]\times[0, \tilde{T}]$ and let $\Omega(\gamma, \tilde{T})$, $\gamma \in (0, \alpha]$, be the set of the functions $U(x, t)$ defined in $R_{\tilde{T}}^{(b)}$ such that $U \in C_{1+\gamma} \cap C_{2,0}$, $U(x, 0) = h^*(x)$, $U_x(0, t) = \psi(t)$, where $h^*(x)$ is a smooth extension of $h(x)$.

For any $U \in \Omega(\gamma, \tilde{T})$ consider the following problem

$$a(x, t, U, V, S)V_{xx} - V_t = Q(x, t, U, U_x, V_x, S) \tag{2.1}$$

in $D_T = \{(x, t): 0 < x < S(t), 0 < t < T\}$,

$$S(0) = b\,, \tag{2.2}$$

$$V(x, 0) = h'(x) \qquad (0 < x < b)\,, \tag{2.3}$$

$$V(0, t) = \psi(t)\,, \tag{2.4}$$

$$V\big(S(t), t\big) = 0 \qquad (0 < t < T)\,, \tag{2.5}$$

$$\dot{S}(t) = \lambda_1(S(t), t, U(S(t), t), V_x(S(t), t)) \qquad (0 < t < T)\,, \tag{2.6}$$

where $Q(x, t, U, U_x, V_x, S) = q_x + q_u U_x + q_p V_x - a_x V_x - a_u U_x V_x - a_p V_x^2$; the arguments of a, q and of their derivatives are $x, t, U, U_x S$.

The free boundary problem (2.1)-(2.6) has been studied in [**2**]$_3$. The assumptions made in § **2.1** are sufficient to ensure the existence of a unique solution (T, S, V) of this problem for any U in $\Omega(\gamma, T)$ (4).

2.3. - Estimates on (T, S, V)

Let us introduce the following subset of $\Omega(\gamma, T)$: $B(K, \gamma, K_1, T) = \{U \in \Omega(\gamma, T): \|U\|_{C_{2,0}} \leqslant K,\ \|U\|_{C_\gamma} \leqslant K_1\}$, with suitably large K, K_1.

For any $U \in B(K, \gamma, K_1, T)$ the following estimates hold true:

$$T \geqslant T_0(K, K_1, \gamma)\,, \tag{2.8}$$

$$\|V\|_{C_{1,0}} \leqslant N_1(K)\,, \tag{2.9}$$

$$\|V\|_{C_{\beta(K)}} \leqslant N_2(K)\,, \tag{2.10}$$

$$\|V\|_{C_{1+\beta(K)}} \leqslant N_3(K)\,, \tag{2.11}$$

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(4) Remark that in [**2**]$_3$ we assumed that the leading coefficient, the source term and the function appearing in the free boundary condition were continuously differentiable (see assumptions (*B*)-(*D*), § **3**). Actually, the assumption $h \in H_{2+\alpha}$ and thm. 5.1 p. 561 of [**3**] allow us to replace the differentiability w.r.t. t by Hölder continuity.

(2.12) $\|V\|_{C_{2,1}}(D^{\tau}_{T_0}) \leqslant N_4(K, K_1, \gamma, \|U\|_{C_{1+\gamma}})\tau^{-\frac{1}{2}}, \quad D^{\tau}_{T_0} = D_{T_0} \cap \{(x, t): t > \tau\}$,

(2.13) $b/2 < S(t) < (3/2)b, \quad t \in (0, T_0)$,

(2.14) $\|S\|_{C_1(0,T_0)} \leqslant N_5(K)$, (2.15) $\|\dot{S}\|_{H_{\beta(K)/2}} \leqslant N_6(K, K_1, \gamma)$,

where the constants $\beta \in (0, 1)$, T_0 and N_i depend also on the data and on the coefficients.

2.4. - Definition of the operator $\mathcal{T}$

Let us define

$$\tilde{U}(x, t) = \int_0^x V(\xi, t)\,d\xi + \int_0^t \{a[0, \tau, U(0, \tau), \psi(\tau), S(\tau)]V_x(0, \tau) \tag{2.16}$$
$$- q[0, \tau, U(0, \tau), \psi(\tau), S(\tau)]\}\,d\tau + h(0), \quad (x, t) \in D_{T_0}$$

and extend it smoothly to $R^{(b)}_{T_0}$ in a fixed way. Because of the estimates of § **2.3** we have

$$\tilde{U} \in \Omega(\beta, T_0). \tag{2.17}$$

The equation

$$\tilde{U} = \mathcal{T} U \tag{2.18}$$

defines the operator $\mathcal{T}: B(K, \gamma, K_1, T_0) \to \Omega(\beta, T_0)$.

2.5. - Solutions of (C1) associated to fixed points of $\mathcal{T}$

If $u \in B(K, \gamma, K_1, T_0)$ is a fixed point of $\mathcal{T}$ the corresponding solution (T_0, s, v) of (2.1)-(2.6) generates the solution (T_0, s, u) of Problem (C1).

As a matter of fact, it is

$$u_x = v, \quad (x, t) \in D_{T_0} \tag{2.19}$$

and equation (2.1) can be written

$$v_t = \frac{\partial}{\partial x}[a(x, t, u, u_x, s)v_x - q(x, t, u, u_x, s)]. \tag{2.20}$$

From (2.16), (2.17) and (2.19), (2.20) it is easy to derive equation (1.1) for u. Also conditions (1.2), (1.3), (1.4), (1.6), (1.9) are easily obtained.

Conversely, it is obvious that if (T_0, s, u) is a solution of Problem (C.1), then u (with the above mentioned extension) is a fixed point of $\mathcal{T}$.

2.6. - Estimates on $\mathcal{T}U$

From § **2.3** and from (2.16) we infer the following estimates on $\mathcal{T}U$

$$\|\mathcal{T}U\|_{C_{1+\beta}} \leqslant N_7(K), \qquad \|\mathcal{T}U\|_{C_{2,0}} \leqslant N_8(K). \tag{2.21}$$

Moreover, estimates of type (2.11), (2.12) apply to $\|\mathcal{T}U\|_{C_{2+\beta}}$ and to $\|\mathcal{T}U\|_{C_{3,1}}$.

We need a more careful estimate of $\|\mathcal{T}U\|_{C_{2,0}}(R_t^{(b)}) = \|\mathcal{T}U\|_{C_{2,0}(D_t)}$, $\forall t \in (0, T_0)$.

To this purpose, we look for an estimate of $\|V\|_{C_{1,0}(D_t)}$, $t \in (0, T_0)$.

The transformation

$$x/S(t) = y\,, \qquad U(S(t)y, t) = \hat{U}(y, t)\,, \qquad V(S(t)y, t) = \hat{V}(y, t)\,, \tag{2.22}$$

$$Q[S(t)y, t, U(S(t)y, t), U_x(S(t)y, t), V_x(S(t)y, t), S(t)] = \hat{Q}(y, t, \hat{U}, \hat{U}_y, \hat{V}_y, S)$$

carries (2.1), (2.3), (2.4), (2.5) into

$$S^{-2}a(Sy, t, \hat{U}, \hat{V}, S)\hat{V}_{yy} + y\dot{S}S^{-1}\hat{V}_y - \hat{V}_t = \hat{Q} \qquad (y, t) \in (0, 1)\times(0, T_0)\,, \tag{2.23}$$

$$\hat{V}(y, 0) = h'(by) \qquad y \in (0, 1)\,, \tag{2.24}$$

$$\hat{V}(0, t) = \psi(t)\,, \tag{2.25}$$

$$\hat{V}(1, t) = 0 \qquad t \in (0, T_0)\,. \tag{2.26}$$

We split $\hat{V}$ into the sum

$$\hat{V} = \hat{V}_1 + \hat{V}_2\,, \tag{2.27}$$

where $\hat{V}_1$ solves

$$L\hat{V}_1 = S^{-2}a(Sy, t, \hat{U}, \hat{V}, S)\hat{V}_{1,yy} - \hat{V}_{1,t} = \hat{Q} - y\dot{S}S^{-1}\hat{V}_y \tag{2.28}$$

with zero initial and boundary conditions, and $\hat{V}_2$ solves

$$L\hat{V}_2 = 0 \tag{2.29}$$

and satisfies (2.24)-(2.26). It is well known that $\hat{V}_1$ and $\hat{V}_2$ exist.

Representing $\hat{V}_1$ by means of the Green function in $(0,1)\times(0,T_0)$ of the parabolic operator L and using standard estimates (see e.g. [**3**], p. 413), as a consequence of (2.9), (2.10), (2.13), (2.14) we get

$$(2.30)\quad \sup_{\substack{0<y<1\\0<\tau<t}} |\hat{V}_1(y,\tau)| \leqslant N_9(K,K_1)\,t\,, \qquad (2.30)'\quad \sup_{\substack{0<y<1\\0<\tau<t}} |\hat{V}_{1,y}(y,\tau)| \leqslant N_9(K,K)\,t^{\frac{1}{2}}.$$

Concerning $\hat{V}_2$, it is obviously dominated as follows

$$(2.31)\qquad \sup_{\substack{0<y<1\\0<\tau<t}} |\hat{V}_2(y,\tau)| \leqslant b\|h''\|_{C(0,b)} + \|\varphi\|_{C_1(0,T_0)}\,t\,,$$

and an estimate for $\hat{V}_{2,y}$ is (see [**2**]$_3$)

$$(2.31)'\qquad \sup_{\substack{0<y<1\\0<\tau<t}} |\hat{V}_{2,y}(y,\tau)| \leqslant b\|h\|_{C(0,b)} + N_{10}(K,K_1)\,t^{\frac{1}{2}}.$$

As a consequence of (2.27), (2.30), (2.30)′, (2.31), (2.31)′ we get

$$(2.32)\qquad \|V\|_{C_{1,0}(D_t)} \leqslant (1+b)\|h''\|_{C(0,b)} + N_{11}(K,K_1)\,t^{\frac{1}{2}}\,.$$

From (2.16) and (2.32) we deduce the desired estimate

$$(2.33)\qquad \|\mathcal{T}U\|_{C_{2,0}(R_t^{(b)})} \leqslant c(b)\|h\|_{C_2(0,b)} + N_{12}(K,K_1)\,t^{\frac{1}{2}}\ (^5)\,.$$

Now, we choose the parameters entering B as follows

$$(2.34)\qquad K=\bar{K}=2c(b)\|h\|_{C_2(0,b)}\,,\qquad \gamma=\bar{\gamma}=\beta(\bar{K})\,,\qquad K_1=\bar{K}_1=N_7(\bar{K})$$

and

$$(2.35)\qquad T=\bar{T}=\min\{T_0,\ [c(b)\|h\|_{C_2(0,b)}/N_{12}(\bar{K},\bar{K}_1)]^2\}\,.$$

We find

$$(2.36)\qquad \|\mathcal{T}U\|_{C_{2,0}(R_{\bar{T}}^{(b)})} \leqslant \bar{K}\,,$$

$$(2.36)'\qquad \|\mathcal{T}U\|_{C_{\bar{\gamma}}(R_{\bar{T}}^{(b)})} \leqslant \bar{K}_1\,,\qquad \forall U\in B(\bar{K},\bar{\gamma},\bar{K}_1,\bar{T})\,.$$

Therefore

$$(2.37)\qquad \mathcal{T}: B(\bar{K},\bar{\gamma},\bar{K}_1,\bar{T}) \to B(\bar{K},\bar{\gamma},\bar{K}_1,\bar{T})\,.$$

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(5) An alternative estimate of the type $C(b)\|h\|_{C_2(0,b)} + N(K)\,t^{\beta(K)/2}$ follows immediately from (2.11).

2.7. - Continuity of $\mathcal{T}$

Take $U^{(1)}$, $U^{(2)} \in B(\bar{K}, \bar{\gamma}, \bar{K}_1, \bar{T})$ and let $(\bar{T}, S^{(1)}, V^{(1)})$, $(\bar{T}, S^{(2)}, V^{(2)})$ be the corresponding solutions of (2.1)-(2.6).

Set $V^{(i)} = 0$ for $x > S^{(i)}(t)$, $i = 1, 2$, and define

(2.38) $\delta(t) = S^{(1)}(t) - S^{(2)}(t)$, (2.39) $W(x, t) = V^{(1)}x, t) - V^{(2)}(x, t)$.

From (2.16) we have

$$(2.40) \qquad \|\mathcal{T} U^{(1)} - \mathcal{T} U^{(2)}\|_{C_{2,0}(R_t^{(b)})}$$

$$\leqslant N\{\sup_{R_t^{(b)}}(|W|+|W_x|) + t[\|\delta\|_{C(0,t)} + \|W_x(0,\cdot)\|_{C(0,t)} + \|U^{(1)}(0,\cdot) - U^{(2)}(0,\cdot)\|_{C(0,t)}]\}\,.$$

The function

$$(2.41) \qquad \hat{W}(y, t) = W(ys(t), t)$$

vanishes for $y = 0$, $y = 1$, $t = 0$, and satisfies the equation

$$(2.42) \qquad L^{(1)}\hat{W} = \hat{\Theta} \qquad (y, t) \in (0, 1)\times(0, \bar{T})\,,$$

where $L^{(1)}$ is the operator defined in (2.28) with $S = S^{(1)}$, $\hat{U} = \hat{U}^{(1)}$, $\hat{V} = \hat{V}^{(1)}$, while the source term $\hat{\Theta}$ can be dominated by

$$Nt^{-\frac{1}{2}}\{\|\hat{U}^1 - \hat{U}^2\|_{C_{1,0}((0,1)\times(0,t))} + \|\hat{W}\|_{C((0,1)\times(0,t))} + \|\delta\|_{C(0,t)}\} + N\{\|\dot{\delta}\|_{C(0,t)} + |\hat{W}_y|\}\,,$$

owing to (2.9)-(2.15).

Here and in the following, constants N depend also on the second order derivatives of a and q (except for $\partial^2/\partial t^2$).

Using the techniques of [**2**]$_3$, Sec. 4, we obtain

$$(2.43) \qquad \|\hat{W}\|_{C_{1,0}((0,1)\times(0,t))} \leqslant Nt^{\frac{1}{2}}\{\|\dot{\delta}\|_{C(0,t)} + \|\hat{U}^{(1)} - \hat{U}^{(2)}\|_{C_{1,0}((0,1)\times(0,t))}\}\ \forall t \in (0, \bar{T}).$$

On the other hand, from theorem 2 of [**2**]$_3$ [6]

$$\|\delta\|_{C_1(0,t)} \leqslant N\{\|\delta\|_{C(0,t)} + \|U^{(1)} - U^{(2)}\|_{C_{1,0}(R_t^{(b)})}\}\,,$$

[6] This theorem can be easily extended to cover the case in which $s(t)$ appears in a and q.

which implies

$$\|\delta\|_{C_1(0,t)} \leqslant N\|U^{(1)} - U^{(2)}\|_{C_{1,0}(R_t^{(b)})}, \tag{2.44}$$

for t in a suitable time interval $(0, T_1) \subset (0, \overline{T})$.

The final estimate resulting from (2.40), (2.43) and (2.44) is

$$\|\mathscr{T}U^{(1)} - \mathscr{T}U^{(2)}\|_{C_{2,0}(R_t^{(b)})} \leqslant Nt^{\frac{1}{2}}\|U^{(1)} - U^{(2)}\|_{C_{1,0}(R_t^{(b)})} \ \forall t \in (0, T_1). \tag{2.45}$$

2.8. - Existence and uniqueness theorem

From (2.45) it follows that there exists $\hat{T} \in (0, T_1)$ such that $\mathscr{T}$ is a contractive mapping of $B(\overline{K}, \bar{\gamma}, \overline{K}_1, \hat{T})$ into itself, with respect to the norm of $C_{1,0}(R_{\hat{T}}^{(b)})$. Since $\mathscr{T}B$ is closed w.r.t. this norm, and $\mathscr{T}^2 B \subset \mathscr{T}B$ the results of **2.5** allow us to state the following theorem

Theorem 2.1. *Under the assumptions listed in* **2.1**, *Problem* (C1) *has a solution* $(\hat{T}, s, u)$, *which is unique in* $(0, \hat{T})$. *Moreover* $s \in H_{1+\bar{\gamma}}[0, \hat{T}]$ *and* $u_x \in C_{1+\bar{\gamma}}(\overline{D}_{\hat{T}})$.

2.9. - Continuous dependence

Let Σ be a set of data and coefficients such that $b_1 \geqslant b \geqslant b_0 > 0$, $\|h\|_{H_{2+\alpha}}$ and $\|\psi\|_{H_{1+\alpha}}$ are uniformly bounded and such that a, q, λ_1 satisfy assumptions (α)-(δ) also in a uniform way. For any element $\sigma = \{b, h, \psi, a, q, \lambda_1\} \in \Sigma$ the a priori estimates on (T, s, u) derived above are uniform and we can define the norm

$$\Delta a = \sup_{\substack{0<x<3b_1/2 \\ 0<t<\overline{T} \\ \|u\|_{C_{1,0}}<N \\ b_0/2<b<3b_1/2}} |a^{(1)} - a^{(2)}|, \tag{2.46}$$

and similarly the norms Δa_x, Δa_u, Δa_p, Δq_v, Δq_x, Δq_u, Δq_p, $\Delta\lambda$, for any pair

$$\sigma_1 = \{b^{(1)}, h^{(1)}, \psi^{(1)}, a^{(1)}, q^{(1)}, \lambda^{(1)}\}, \qquad \sigma_2 = \{b^{(2)}, h^{(2)}, \psi^{(2)}, a^{(2)}, q^{(2)}, \lambda^{(2)}\}.$$

Using techniques similar to those employed in § **2.7** it can be proved that for any $U \in B(\overline{K}, \bar{\gamma}, \overline{K}_1, \hat{T})$, $\mathscr{T}U$ depends continuously on $\sigma \in \Sigma$ in a uniform way, with respect to the distance

$$(2.47)\qquad \varrho(\sigma_1, \sigma_2) = |b^{(1)} - b^{(2)}| + \|h^{(1)} - h^{(2)}\|_{C_1} + \|\psi^{(1)} - \psi^{(2)}\|_{C_2}$$

$$+ \Delta a + \Delta a_x + \Delta a_p + \Delta q + \Delta q_x + \Delta q_u + \Delta q_p + \Delta\lambda$$

(set $h^{(i)} = 0$ for $x > b^{(i)}$). More precisely

$$(2.48)\qquad \|\mathscr{T}_{\sigma_1} U - \mathscr{T}_{\sigma_2} U\|_{C_{2,0}(R_{\hat{T}}^{(b)})} \leqslant N\varrho(\sigma_1, \sigma_2) .$$

As a consequence (see [4], p. 630), the following theorem is proved

Theorem 2.2. *For any pair* σ_1, $\sigma_2 \in \Sigma$ *the corresponding solutions* $(\hat{T}, s^{(1)}, u^{(1)})$, $(\hat{T}, s^{(2)}, u^{(2)})$ *satisfy the inequalities*

$$(2.49)\qquad \|u^{(1)} - u^{(2)}\|_{C_{2,0}} \leqslant N\varrho(\sigma_1, \sigma_2) ,$$

$$(2.50)\qquad \|s^{(1)} - s^{(2)}\|_{C_1} \leqslant N\varrho(\sigma_1, \sigma_2) .$$

Note that (2.50) follows from (2.49) and (2.44).

3. - Problem (C2)′

The proof of well-posedness of Problem (C2)′ follows the general scheme of § **2**, although many nontrivial modifications are needed. The main differences will be in the definition of the operator $\mathscr{T}$ and in the choice of the functional spaces to be used.

3.1. - Assumptions

In addition to (α)-(δ) of § **2.1** we shall assume

(α′) $h \in H_{3+\alpha}[0, b]$, $\varphi \in H_{2+\alpha}[0, \tilde{T}]$;

(β′ − γ′) a, q are independent of s (and differentiable w.r.t. t),

$$(3.1)\qquad a(0, 0, h(0), h'(0))\, h''(0) - q(0, 0, h(0), h'(0)) = \dot{\varphi}(0) ,$$

$$(3.2)\qquad a(b, 0, 0, 0)\, h''(b) - q(b, 0, 0, 0) = 0 ,$$

and satisfying standard growing conditions w.r.t. u, u_x (see e.g. thm. 5.2, p. 564 of [**3**]);

(δ') λ_2 is continuously differentiable and

$$|\lambda_2(x, t, u, p, \vartheta)| \leq \nu(|u|, |p|, |\vartheta|)$$

for some continuous nondecreasing function ν.

The independence of a and q of s has been introduced only for the sake of simplicity.

3.2. - Auxiliary free-boundary problem

Let $\Omega_1(\gamma, \tilde{T})$ be the set of the functions U defined in $R_{\tilde{T}}^{(b)}$ such that

$$U \in C_{2+\gamma}, \qquad U(0, t) = \varphi(t), \qquad U(x, 0) = h(x)$$

(with a smooth extension of $h(x)$ for $x > b$).

For any $U \in \Omega_1(\gamma, \tilde{T})$ we introduce the function

$$\chi(x, t, U, U_x, Z, Z_x) = q_t + q_u Z + q_p Z_x - (a_t + a_u Z + a_p Z_x)(Z + q)/a, \tag{3.3}$$

where the arguments of a, q and of their derivatives are x, t, U, U_x.

Then we define a differentiable function $\chi^*(x, t, U, U_x, Z, Z_x)$ such that

$$\chi^* = \chi \quad \text{for } |Z| \leq Z_0, \tag{3.4}$$

$$|\chi^*(x, t, u, p, Z, 0)| \leq \sup_{|\vartheta| \leq Z_0 + 1} |\chi(x, t, u, p, \vartheta, 0)|, \qquad Z > Z_0 + 1, \tag{3.4$'$}$$

for some given $Z_0 > 0$, and we consider the following free boundary problem

$$a(x, t, U, U_x) Z_{xx} - Z_t = \chi^*(x, t, U, U_x, Z, Z_x) \quad \text{in } D_T, \tag{3.5}$$

$$S(0) = b, \tag{3.6}$$

$$Z(x, 0) = a(x, 0, h, h') h''(x) - q(x, 0, h, h'), \qquad 0 < x < b, \tag{3.7}$$

$$Z(0, t) = \dot{\varphi}(t), \qquad 0 < t < T, \tag{3.8}$$

(3.9) $$Z(S(t), t) = 0\,, \quad 0 < t < T\,,$$

(3.10) $$\dot{S}(t) = \lambda_2(S(t), t, U(S(t), t), U_x(S(t), t), Z_x(S(t), t))\,, \quad 0 < t < T\,.$$

A unique solution (T, S, Z) to (3.5)-(3.10) exists under the assumptions listed above (see [**2**]$_3$).

3.3. - Estimates on (T, S, Z)

Let $B_1(K, \gamma, K_1, T)$, $T \leqslant \tilde{T}$, be the subset of $\Omega_1(\gamma, \tilde{T})$ such that

(3.11) $$\|U\|_{C_{2,0}(R_T^{(b)})} \leqslant K\,, \qquad \|U\|_{C_{1+\gamma}(R_T^{(b)})} \leqslant K_1\,.$$

If $U \in B_1(K, \gamma, K_1, T)$, from [**2**]$_2$ we have

(3.12) $$T \geqslant T_0(K, \gamma, K_1, Z_0)\,,$$ (3.13) $$\|Z\|_{C_{1+\beta(K)}} \leqslant M_1(K, Z_0)\,,$$

(3.14) $$\|S\|_{H^{1+\beta(K)}} \leqslant M_2(K, \gamma, K_1, Z_0)\,,$$

besides a nonuniform estimate of $\|Z\|_{C_{2,1}}$ similar to (2.12). Like the constants N_i in § **2**, also the constants M_i are obviously dipendent on the data and the coefficients.

3.4. - Definition of the operator $\mathscr{T}_1$

For any $U \in B_1(K, \gamma, K_1, T_0)$ we define

(3.15) $$\mathscr{T}_1 U = \tilde{U}\,,$$

where $\tilde{U}$ is the solution of the following nonlinear initial-boundary value problem

(3.16) $$a(x, t, \tilde{U}, \tilde{U}_x)\tilde{U}_{xx} - \tilde{U}_t = q(x, t, \tilde{U}, \tilde{U}_x)\,, \qquad \text{in } D_{T_0}\,,$$

(3.17) $$\tilde{U}(x, 0) = h(x)\,, \quad 0 < x < b\,,$$

(3.18) $$\tilde{U}(0, t) = \varphi(t)\,, \quad 0 < t < T_0\,,$$

(3.19) $$\tilde{U}(S(t), t) = h(b) + \int_0^t \dot{S}(\tau)\, U_x(S(\tau), \tau)\, d\tau\,, \qquad 0 < t < T_0\,.$$

Owing to the compatibility conditions assumed for h and φ at the points $(0, 0)$, $(b, 0)$, this problem has a unique solution in $C_{2+\beta}(\overline{D}_{T_0})$ (Thm. 5.2, p. 564 of [**3**]). In (3.15) we mean that U is extended smoothly to $R^{(b)}_{T_0}$ in a prescribed way. Therefore

$$\mathscr{T}_1 \colon B_1(K, \gamma, K_1, T_0) \to \Omega_1(\beta, T_0) . \tag{3.20}$$

3.5. - Estimates on $\mathscr{T} U$

The norm $\|\tilde{U}\|_{C_{2+\beta}}$ is estimated in terms of the data and the coefficients and of $\|S\|_{H_{1+\beta}}$, $\|U\|_{C_{1+\beta}}$, i.e.

$$\|\tilde{U}\|_{C_{2+\beta(K)}} \leqslant M_3(K, \gamma, K_1, Z_0) , \tag{3.21}$$

where (3.14) has been used.

Estimate (3.21) implies

$$\|\tilde{U}\|_{C_{2,1}} \leqslant M_0 + M_4(K, \gamma, K_1, Z_0) t^{\beta(K)} , \tag{3.22}$$

with M_0 independent of K, γ, K_1, Z_0.

Moreover (see theorem 5.1, p. 561 of [**3**])

$$\|\tilde{U}\|_{C_{1+\beta(K)}} \leqslant M_5(K) . \tag{3.23}$$

Taking e.g. $\overline{K} = 2M_0$, $\overline{K}_1 = M_5(\overline{K})$, $\bar{\gamma} = \beta(\overline{K})$ from (3.22), (3.23) we can find $T_1 \leqslant T_0(\overline{K}, \bar{\gamma}, \overline{K}_1, Z_0)$ such that

$$\|\tilde{U}\|_{C_{2,0}(R^{(b)}_{T_1})} \leqslant \overline{K} , \qquad \|\tilde{U}\|_{C_{1+\bar{\gamma}}(R^{(b)}_{T_1})} \leqslant \overline{K}_1 . \tag{3.24}$$

At this point we remark that (3.3) and the maximum principle yield

$$|Z(x, t)| \leqslant M_6(K)\Big\{1 + \int_0^t \sup_{x \in (0, s(\tau))} |Z(x, \tau)|^2 \mathrm{d}\tau\Big\};$$

hence for any sufficiently large Z_0 we can calculate a $T_2(K)$ such that

$$|Z(x, t)| \leqslant Z_0 , \qquad 0 \leqslant x \leqslant S(t) , \qquad 0 \leqslant t \leqslant T_2(K) . \tag{3.25}$$

Therefore, setting $\overline{T} = \min\big(T_1, T_2(\overline{K})\big)$ we conclude that

$$\mathscr{T}_1 \colon B_1(\overline{K}, \bar{\gamma}, \overline{K}_1, \overline{T}) \to B_1(\overline{K}, \bar{\gamma}, \overline{K}_1, \overline{T}) \tag{3.26}$$

and that χ^* can be identified with χ in (3.5).

3.6. - Solutions of (C2)′ associated to fixed points of $\mathscr{T}_1$

Let u be a fixed point of $\mathscr{T}_1$ and denote by (T_0, s, z) the corresponding solution of (3.5)-(3.10).

From (3.19) we obtain $u_t(s(t), t) = 0$. Hence it is easy to see that the difference $z - u_t$ can be thought as the solution of a linear homogeneous parabolic equation with square summable coefficients and with zero initial and boundary data.

Therefore

$$u_t = z \quad \text{in } D_{T_0}, \tag{3.27}$$

which shows that (T_0, s, u) solves (C2′).

3.7. - Existence and uniqueness theorem

Using basically the same methods of Sec. 4 of [**3**], it can be seen that for the solutions $\tilde{U}^{(1)}$, $\tilde{U}^{(2)}$ of (3.16)-(3.19) corresponding to two respective elements $U^{(1)}, U^{(2)} \in B_1(\bar{K}, \bar{\gamma}, \bar{K}_1, \bar{T})$, we have

$$\|\tilde{U}^1 - \tilde{U}^2\|_{C_{1,0}(R_t^{(b)})} \leqslant M(\bar{K}) t^{\frac{1}{2}} \|U^{(1)} - U^{(2)}\|_{C_{1,0}(R_t^{(b)})}, \ \forall t \in (0, \bar{T}) . \tag{3.28}$$

Thus the operator $\mathscr{T}_1$ is a contractive mapping w.r.t. the norm of $C_{1,0}(R_{\hat{T}}^{(b)})$ for some $\hat{T} \in (0, \bar{T}]$. The set $\mathscr{T}_1^2 B_1 \subset \mathscr{T}_1 B_1$ is closed w.r.t. such a norm and the existence of a unique solution $(\hat{T}, s, u)$ of (C2′) is proved.

Theorem 3.1. *Under the assumptions listed in* § **3.1**, *Problem* (C.2)′ *has a solution* $(\hat{T}, s, u)$, *which is unique in* $(0, \hat{T})$. *Moreover* $s \in H_{1+\bar{\gamma}}[0, \hat{T}]$ *and* $u_{xx} \in C_{1+\bar{\gamma}}(\bar{D}_{\hat{T}})$.

3.8. - Continuous dependence

Let Σ be the set whose elements $\sigma = \{b, h, \varphi, a, q, \lambda_2\}$ satisfy all the assumptions listed in **3.1** in a uniform way. For any $\sigma \in \Sigma$ we have a solution $(\hat{T}, s, u)$ of (C2′) and a uniform estimate on the norm of u in $C_{1,0}(R_{\hat{T}}^{(b)})$ is available. This allows us to define the quantities Δa, Δq, Δa_t, Δq_t, ..., $\Delta\lambda_2$ as in **2**. Then we can state

Theorem 3.2. *For any pair* σ_1, $\sigma_2 \in \Sigma$ *the corresponding solutions* $(\hat{T}, s^{(1)}, u^{(1)})$, $(\hat{T}, s^{(2)}, u^{(2)})$ *satisfy the inequalities*

$$\|u^{(1)} - u^{(2)}\|_{C_{1,0}(R_{\hat{T}}^{(b)})} \leq N\varrho(\sigma_1, \sigma_2) \,, \tag{3.29}$$

$$\|s^{(1)} - s^{(2)}\|_{C_1(0,\hat{T})} \leq N\varrho(\sigma_1, \sigma_2) \,, \tag{3.30}$$

with

$$\varrho(\sigma_1, \sigma_2) = |b_1 - b_2| + \|h^{(1)} - h^{(2)}\|_{C_3} + \|\varphi^{(1)} - \varphi^{(2)}\|_{C_2}$$
$$+ \Delta a + \Delta a_t + \Delta a_u + \Delta a_p + \Delta q + \Delta q_t + \Delta q_u + \Delta q_p + \Delta \lambda_2 \,.$$

The proof of Theorem 3.2 is omitted for the sake of brevity.

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