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An evolution problem arising from a stochastic population model with nonlinear birth and death parameters (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

The theory of semigroups of linear bounded transformations was used in [1] to study an evolution problem arising from stochastic population theory. Existence and uniqueness of a positive and norm invariant solution were proved, assuming that the birth and death parameters are linearly dependent on the population size.

Such an assumption may not be realistic in a physical phenomenon such as a population growth. In this paper, we examine a more general « birth and death » process, in which the parameters are not linearly dependent on the population size. We prove that the initial value problem has a unique positive and norm invariant solution belonging to the Banach space of all summable sequences.

Let $P(n, t)$ be the probability that, at time t, a given population (e.g., a population of bacteria in a culture) is composed of n individuals. It is known that a general method to derive the distribution of the population size at time t

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makes use of the Chapman-Kolmogorov equations [4]. Following [2], a general « birth and death » process leads to the study of the system

$$(1)\qquad \frac{\partial P(n,t)}{\partial t} = -[p(n)+q(n)]n\,P(n,t) + p(n-1)[n-1]P(n-1,t) + q(n+1)[n+1]P(n+1,t) \qquad (t>0,\ n=0,1,2,\dots),$$

with the initial conditions

$$(2)\qquad P(n,0) = P_0(n) \qquad (n=0,1,2,\dots).$$

In (1) and (2), $P(-1,t)\equiv 0$ and the $P_0(n)$ are given so that

$$(3)\qquad 0 \leqslant P_0(n) \leqslant 1\,, \quad \sum_{n=0}^{\infty} P_0(n) = 1\,.$$

Moreover, the parameters p and q are assumed to be nonnegative functions of the number n of individuals, and such that

$$(4)\qquad 0 \leqslant p_1 \leqslant p(n) \leqslant p_2 < \infty\,, \quad 0 \leqslant q_1 \leqslant q(n) \leqslant q_2 < \infty\,, \qquad (n=0,1,2,\dots)$$

where p_1, q_1, p_2, q_2 are given constants.

Note that system (1) can be derived under the assumption that, for a given population size, births and deaths occur independent of each other. Also, if the population size is n at time t, the probabilities of birth events and of death events during the time interval Δt are respectively $p(n)nP(n,t)\Delta t$ and $q(n)nP(n,t)\Delta t$.

To explain the « physical » meaning of system (1), we note that the probability $P(n,t)$ of having a population of n individuals at time t is increased by birth events in a population of $(n-1)$ individuals (see the term $p(n-1)\cdot[n-1]P(n-1,t)$), and is decreased by birth events in a population of n individuals (see the term $-p(n)nP(n,t)$). A similar explanation holds for death events.

Finally, the initial probabilities $P_0(n)$ must belong to $[0,1]$ and $\sum_{n=0}^{\infty} P_0(n) = 1$, in agreement with their physical meaning.

2. - Definitions and preliminary remarks

Let $X = l^1$ be the real Banach space of all summable sequences of real numbers $f = \{f(n),\ n=0,1,2,\dots\}$ with norm $\|f\| = \sum_{n=0}^{\infty} |f(n)|$ and let X^+ be the closed positive cone of X: $X^+ = \{f : f \in X;\ f(n) \geqslant 0,\ n=0,1,2,\dots\}$.

System (1) suggests us to introduce the auxiliary operators H, defined by the diagonal part of system (1), and K, defined by the non-diagonal part.

$$(5)\qquad [Hf]_n = [p(n) + q(n)]nf(n) \qquad (n = 0, 1, 2, \ldots),$$

$$D(H) = \{f : f \in X;\ \sum_{n=0}^{\infty} [p(n) + q(n)]n|f(n)| < \infty\},$$

$$(6)\qquad [Kf]_n = p(n-1)[n-1]f(n-1) + q(n+1)[n+1]f(n+1) \quad (n = 1, 2, \ldots),$$

$$[Kf]_0 = q(1)f(1),$$

$$D(K) = D(H),$$

where $[Hf]_n$ denotes the $(n+1)$-th component of the element $Hf \in X$. Further, let us define the following operator

$$(7)\qquad [Af]_n = -[Hf]_n + [Kf]_n, \qquad D(A) = \{f : f \in X;\ \sum_{n=0}^{\infty} |[Af]_n| < \infty\}.$$

Note that $D(H) \subset D(A)$, because, if $f \in D(H)$, then

$$\sum_{n=0}^{\infty} |[Af]_n| \leqslant 2 \sum_{n=0}^{\infty} \big(p(n) + q(n)\big) n |f(n)| < \infty$$

and so $f \in D(A)$. Thus, if we consider the operator $(-H + K)$ with domain $D(-H + K) = D(H) \cap D(K) = D(H)$, then $-H + K \subset A$. Moreover, let D_0 be the linear manifold spanned by the canonical base of X, i.e., let D_0 be composed of all the elements of X with a finite number of nonzero components. We define the operator A_0 as the restriction of A with domain $D(A_0) = D_0$.

Following [**3**]$_1$, we finally put

$$(8)\qquad A_r = -H + rK, \qquad D(A_r) = D(H),$$

where r is a real parameter, such that $0 \leqslant r < 1$. A_r is, in some sense, an operator «approximating» A, because, if $f \in D(H)$, we have $\|Af - A_r f\| = (1 - r)\|Kf\| \to 0$, as $r \to 1^-$. However, this will become completely clear in the following sections.

3. - Properties of H and K

In this section, we shall investigate some relevant properties of the operator H and K. We have

Lemma 1. (a) *H is densely defined in X*; (b) *for every $z > 0$, $(zI + H)^{-1} \in \mathscr{B}(X)$ with $\|(zI + H)^{-1}g\| \leqslant z^{-1}\|g\|$, $\forall g \in X$*; (c) *H maps $D(H) \cap X^+$ into X^+ and $(zI + H)^{-1}$ maps X^+ into itself $\forall z > 0$*; (d) *for every $z > 0$, $H(zI + H)^{-1} \in \mathscr{B}(X)$, with $\|H(zI + H)^{-1}g\| \leqslant \|g\|$, $\forall g \in X$.*

Proof. (a) Is obvious because D_0 is dense in X and $D_0 \subset D(H)$.

(b) We have directly from the equation $(zI + H)f = g$, $g \in X$, $z > 0$: $f(n) = \big(z + (p(n) + q(n))\,n\big)^{-1} g(n)$, and $f \in D(H)$ because

$$\sum_{n=0}^{\infty} [p(n) + q(n)]\,n\,|f(n)| = \sum_{n=0}^{\infty} \frac{[p(n) + q(n)]n}{z + [p(n) + q(n)]n}\,|g(n)| < \sum_{n=0}^{\infty} |g(n)| < \infty\,.$$

Moreover, $\|f\| = \|(zI + H)^{-1}g\| \leqslant z^{-1}\|g\|$.

(c) If $g \in D(H) \cap X^+$, then

$$[Hg]_n = [p(n) + q(n)]ng(n) \geqslant 0 \qquad (n = 0, 1, 2, \ldots)\,,$$

$$[(zI + H)^{-1}g]_n = \big(z + (p(n) + q(n))\,n\big)^{-1} g(n) \geqslant 0 \qquad (n = 0, 1, 2, \ldots)$$

and (c) is proved.

(d) It follows from (b) that the operator $H(zI + H)^{-1}$ has domain X. Now, $H(zI + H)^{-1}g = g - z(zI + H)^{-1}g$, $\forall g \in X$, $z > 0$, and, if $\varphi \in X^+$, we obtain from the preceding equality

$$[H(zI + H)^{-1}\varphi]_n = \frac{[p(n) + q(n)]n}{z + [p(n) + q(n)]n}\,\varphi_n \geqslant 0\,,$$

$$\|H(zI + H)^{-1}\varphi\| = \sum_{n=0}^{\infty} \frac{[p(n) + q(n)]n}{z + [p(n) + q(n)]n}\,\varphi_n \leqslant \sum_{n=0}^{\infty} \varphi_n = \|\varphi\|\,.$$

Finally, if $f \in X$, let

$$f^-(n) = -f(n) \quad \text{if } f(n) < 0\,, \qquad f^-(n) = 0 \quad \text{if } f(n) \geqslant 0\,,$$

$$f^+(n) = f(n) \quad \text{if } f(n) \geqslant 0\,, \qquad f^+(n) = 0 \quad \text{if } f(n) < 0\,.$$

Then, $f = f^+ - f^-$, $f^+ \in X^+$, $f^- \in X^+$, $\|f\| = \|f^+\| + \|f^-\|$ and so,

$$\|H(zI + H)^{-1}f\| \leqslant \|f^+\| + \|f^-\| = \|f\|\,.$$

Lemma 2. (a) *K maps* $D(H) \cap X^+$ *into* X^+; (b) $\|Kf\| \leqslant \|Hf\|$, $\forall f \in D(H)$; (c) $\|Hf\| = \|Kf\|$, $\forall f \in D(H) \cap X^+$.

Proof. (a) If $g \in D(H) \cap X^+$, we have

$$[Kg]_n = p(n-1)[n-1]g(n-1) + q(n+1)[n+1]g(n+1) \geqslant 0$$

and so, $Kg \in X^+$.

(b) We now have from the definitions (5) and (6)

$$\begin{aligned}\|Kf\| &\leqslant \sum_{n=1}^{\infty} p(n-1)[n-1]|f(n-1)| + \sum_{n=0}^{\infty} q(n+1)[n+1]|f(n+1)| \\ &\leqslant \sum_{n=0}^{\infty} \big(p(n) + q(n)\big) n |f(n)| = \|Hf\|, \qquad \forall f \in D(H).\end{aligned}$$

(c) Immediately follows from the previous relation, with $f \in D(H) \cap X^+$.

Lemma 3. *The linear operator* $F(z) = K(zI + H)^{-1}$, $z > 0$, *has the properties*: (a) $F(z) \in \mathscr{B}(X)$, $\|F(z)f\| \leqslant \|f\|$, $\forall z > 0$, $f \in X$; (b) $F(z)$ *maps* X^+ *into itself*.

Proof. The operator $F(z)$ is defined over the whole space X, because the range of $(zI + H)^{-1}$ is $D(H) = D(K)$.

Finally, (a) and (b) are easily proved by means of Lemmas 1 and 2.

Remark. The proofs of the preceding lemmas are based upon some results obtained by Kato, [**3**]$_1$; see also [**1**].

4. - The abstract problem

System (1)-(3) leads to the following initial-value problem in the Banach space X

$$\text{(9)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t), \quad (t > 0), \quad X - \lim_{t \to 0^+} u(t) = u_0,$$

where A is defined by (7), $u(t) = \begin{pmatrix} P(0, t) \\ P(1, t) \\ \cdots \end{pmatrix}$ is a map from $[0, +\infty)$ into X,

$u_0 = \begin{pmatrix} P_0(0) \\ P_0(1) \\ \cdot\cdot \end{pmatrix}$, and d/dt is a strong derivative, [**3**]$_2$.

By using definition (8), we also consider the «approximating» initial-value problem

$$\text{(10)} \qquad \frac{d}{dt} w(t; r) = A_r w(t; r), \quad (t > 0); \quad X - \lim_{t \to 0^+} w(t; r) = u_0,$$

where $r \in [0, 1)$.

As far as the «approximating» problem is concerned, we obtain from Lemmas 1, 2 and 3:

Lemma 4. (a) *The «approximating» operator* $A_r \in \mathscr{G}(1, 0; X)$, $0 \leqslant r < 1$, [**3**]$_2$; (b) *the semigroup* $\{Z_r(t) = \exp(tA_r), t \geqslant 0\}$, *generated by* A_r, *maps* X^+ *into itself, for every* $t \geqslant 0$.

The preceding lemma can be proved without difficulty, by means of procedures similar to those used in [**3**]$_1$.

Let us now return to the «approximating» problem (10). We can state the following theorem.

Theorem 1. *The «approximating» initial-value problem* (10), *with* $r \in [0, 1)$, *has the unique strict solution* $w(t; r) = Z_r(t) u_0 \in D(H)$, $(t \geqslant 0)$ *if* $u_0 \in D(H)$, *and* $w(t; r) \in D(H) \cap X^+$, *if* $u_0 \in D(H) \cap X^+$.

We are now in position to define the operator $Z(t)$

$$\text{(11)} \qquad Z(t) f = X - \lim_{r \to 1^-} Z_r(t) f, \qquad t \geqslant 0, \quad f \in X,$$

whose relevant properties are summarized in the following theorem, [**3**]$_1$.

Theorem 2. (a) $Z(t)$ *is a semigroup such that* $\|Z(t) f\| \leqslant \|f\|$, $t \geqslant 0$, $f \in X$; (b) $Z(t) g \in X^+$, $\forall g \in X^+$ *and* $\|Z(t) g\| = \|g\|$, $\forall g \in X^+$; (c) *the limit* (11) *holds uniformly with respect to t in each finite interval* $[0, t_0]$; (d) *if G is the generator of* $Z(t)$, *then* $A_0 \subset -H + K \subset G \subset A$.

Remark. Note that G is the smallest extension of A_0 (and also of $-H + K$). In [**3**]$_1$ is proved that, if $\{Z'(t), t \geqslant 0\}$ is a semigroup such that $Z'(t) \geqslant 0$ and its generator G' is an extension of A_0, then $Z'(t) \geqslant Z(t)$, $t \geqslant 0$.

If we now consider the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = Gv(t)\,, \qquad t>0\,; \qquad X - \lim_{t\to 0^+} v(t) = u_0 \in D(G)\,, \tag{12}$$

then, by using Theorem 2, system (12) admits one and only one solution of the form

$$v(t) = \exp\,(Gt)\,u_0 \qquad\qquad u_0 \in D(G),\ (t \geqslant 0)\,. \tag{13}$$

On the other hand, if we assume that $u_0 \in D(H) \cap X^+ \subset D(G) \cap X^+$, then the solution $v(t) \in D(G) \cap X^+ \subset D(A) \cap X^+$ and

$$\|v(t)\| \leqslant \|u_0\| \qquad\qquad \forall t \geqslant 0\,. \tag{14}$$

Since G is a restriction of A, $Gv(t) = Av(t)$, $\forall t \geqslant 0$, and so $v(t)$ also satisfies the initial-value problem (9). We conclude with the following theorem.

Theorem 3. *If* $u_0 \in D(H) \cap X^+$, *the initial-value problem* (9) *has the strict solution* $u(t) = v(t) = \exp\,(Gt)\,u_0$. *Such a solution is the* X*-limit as* $r \to 1^-$ *of the strict solution of the «approximating» initial-value problem* (10). *Moreover* $\|u(t)\| \leqslant \|u_0\|$, $\forall t \geqslant 0$.

5. - Preservation of the norm of $v(t)$

In this section, we shall prove that

$$\|\exp\,(Gt)f\| = \|f\|\,, \qquad\qquad f \in X^+ \qquad\qquad (t \geqslant 0)\,. \tag{15}$$

To this aim, we introduce the space X^* of all bounded linear forms on X

$$X^* = \{f^*\colon f^* = \{f^*(n),\ n = 0, 1, 2, \ldots\},\ \|f^*\| < \infty\}\,,$$

with norm $\|f^*\| = \sup\,\{|f^*(n)|,\ n = 0, 1, 2, \ldots\}$.

If $f \in D_0$, then we have

$$(f^*, A_0 f) = \sum_{n=0}^{\infty} \big(f^*(n), \, -[p(n) + q(n)]nf(n) + p(n-1)[n-1]f(n-1) + q(n+1)[n+1]f(n+1)\big) \tag{16}$$

$$= -\sum_{n=0}^{\infty} np(n)f^*(n)f(n) - \sum_{n=0}^{\infty} nq(n)f^*(n)f(n) + \sum_{m=-1}^{\infty} mp(m)f^*(m+1)f(m) + \sum_{m=1}^{\infty} mq(m)f^*(m-1)f(m)$$

$$= \sum_{n=0}^{\infty} \{-n[p(n)+q(n)]f^*(n)f(n) + np(n)f^*(n+1)f(n) + nq(n)f^*(n-1)f(n)\} .$$

We are now in position to find the adjoint A_0^* of the operator A_0, where $D(A_0^*)$ is composed of all $f^* \in X^*$ for which a $g^* \in X^*$ exists with $(f^*, A_0 f) = (g^*, f)$, for all $f \in D_0$.

Using of (16), we obtain

$$[A_0^* f^*]_n = g^*(n) , \qquad D(A_0^*) = \{f^* : f^* \in X^*, \ g^* \in X^*\} ,$$

where

$$g^*(n) = -n[p(n)+q(n)]f^*(n) + np(n)f^*(n+1) + nq(n)f^*(n-1) \tag{17}$$

provided that $\sup \{|g^*(n)|, \ n = 0, 1, 2, \ldots\} < \infty$.

Now, the semigroup $\exp(Gt)$ satisfies (15), if, for some $\lambda > 0$, the equation

$$(\lambda I - A_0^*)f^* = 0 \tag{18}$$

has no solution $f^* \neq 0$, $[\mathbf{3}]_1$.

By taking into account (17), the equation (18) may be put into the form

$$\lambda f^*(n) + n[p(n)+q(n)]f^*(n) - np(n)f^*(n+1) - nq(n)f^*(n-1) = 0 , \tag{19}$$

where $n = 0, 1, 2, \ldots$. It follows from (19) that $f^*(0) = 0$; furthermore, we obtain, for $n = 1, 2, \ldots$

$$\{\lambda + n(p(n)+q(n))\}f^*(n) = np(n)f^*(n+1) + nq(n)f^*(n-1) ,$$

where $\|f^*\| = \sup \{|f^*(n)|, \ n = 1, 2, \ldots\} < \infty$.

Since λ is a positive number, we have

$$(20)\qquad \begin{aligned} f^*(1) &= [\lambda + p(1) + q(1)]^{-1} p(1) f^*(2)\,, \\ \varphi^*(n) &= [\lambda + np(n) + nq(n)]^{-1}\{p(n)[n+1]\varphi^*(n+1) \\ &\qquad + q(n)[n-1]\varphi^*(n-1)\}\,, \end{aligned}$$

where $\varphi^*(n) = f^*(n)/n$, $n = 2, 3, \ldots$.

If system (19) has a nontrivial solution $f^* \in X^*$, it is possible to find an integer $\bar{n}$ such that

$$(21)\qquad \mu = \sup\{|\varphi^*(n)|,\ n = 2, 3, \ldots\} = |\varphi^*(\bar{n})|\,.$$

By using (21), we have from (20)

$$|\varphi^*(n)| \leqslant \frac{np(n) + nq(n) + p(n) - q(n)}{\lambda + np(n) + nq(n)}\,\mu\,,$$

with $n = 2, 3, \ldots$; and in particular for $n = \bar{n}$

$$(22)\qquad \mu \leqslant \frac{\bar{n}p(\bar{n}) + \bar{n}q(\bar{n}) + p(\bar{n}) - q(\bar{n})}{\lambda + \bar{n}p(\bar{n}) + \bar{n}q(\bar{n})}\,\mu\,.$$

First, consider the case $p_2 > q_1$ (see [**1**]).

Then, $\lambda > p_2 - q_1$ implies that

$$(23)\qquad \frac{\bar{n}p(\bar{n}) + \bar{n}q(\bar{n}) + p(\bar{n}) - q(\bar{n})}{\lambda + \bar{n}p(\bar{n}) + \bar{n}q(\bar{n})} < 1$$

and so inequality (22) leads to $\mu = 0$. Secondly, consider the case $p_2 \leqslant q_1$. Then $p(\bar{n}) - q(\bar{n}) \leqslant 0$ and $\mu = 0$, because λ is a positive number. Hence, if p_1, q_1, p_2, q_2 are given (see (4)), it is clear that the system (20) has only the trivial solution $\varphi^*(n) = 0$ if $\lambda > |p_2 - q_1|$ and so $f^*(n) = 0$, for $n = 2, 3, \ldots$. On the other hand, also $f^*(0) = 0$ and $f^*(1) = 0$. Hence, equation (18) has only the trivial solution, provided that $\lambda > |p_2 - q_1|$.

We can now use theorem 3 of [**3**]$_1$ and conclude that $\|\exp(Gt)f\| = \|f\|$, $f \in X^+$, $t \geqslant 0$.

Moreover, $\exp(Gt)$ is the only semigroup whose generator is an extension of A_0.

Let us now discuss briefly some consequences of (15). If $u_0 \in D_0 \cap X^+$ and $\|u_0\| = \sum_{n=0}^{\infty} P_0(n) = 1$, then the solution $u(t) = \exp(Gt)u_0$ of the initial-value problem (9) is such that

$$\|u(t)\| = \|\exp(Gt)\,u_0\| = \|u_0\| = 1 \qquad (t \geqslant 0),$$

where $u(t) = \{P(0, t), P(1, t), \ldots, P(n, t), \ldots\} \in X^+$.

Since $P(n, t)$ are probabilities, the meaning of the relation $\|u(t)\| = \sum_{n=0}^{\infty} P(n, t) \equiv 1$ is obvious.

6. - Concluding remark

If it is assumed a deterministic law of growth for a « birth and death » process, then the corresponding population size at time t is the expected value of population size under stochastic assumptions. Therefore, it is of some interest to study the first moment of the population under consideration.

Now, the first moment is defined by

$$N(t) = \langle n \rangle(t) = \sum_{n=0}^{\infty} nP(n, t)$$

and is the expected value (average number) of individuals. If the parameters p and q in the system (1) are independent on n, i.e. they are given constants, then it is possible to derive in a rigorous way the evolution equation for the first moment $N(t)$, (see [**1**]).

Unfortunately, in our case, because of the nonlinear nature of the birth and death parameters, it is impossible to find a similar equation.

In any case, it is possible, for large population sizes, to approximate the stochastic process by a deterministic process plus a noise term. However, such an approximation is beyond our purposes.

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Sunto

Si studia un problema ai valori iniziali relativo ad un modello di crescita di una popolazione in cui i parametri di nascita e morte dipendono non linearmente dal numero di individui. Si prova l'esistenza e l'unicità di una soluzione positiva ed invariante in norma, appartenente allo spazio di Banach delle successioni sommabili, facendo uso della teoria dei semigruppi di operatori lineari.

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