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On the almost periodic solution to some parabolic quasi-variational inequalities (**)

A Giorgio Sestini per il suo 70° compleanno

1. - Introduction and results

The problem of existence and uniqueness of an almost periodic solution to a parabolic quasi-variational inequality has been treated exhaustively in the case of coefficients independent on the time, $[\mathbf{3}]_{1,2,3}$, and such a treatement can be extended to the case of coefficients depending almost periodically on the time.

The aim of this paper is to give an existence-uniqueness result for the almost periodic solution of the parabolic quasi-variational inequalities treated by A. Bensoussan, J. L. Lions [**1**].

Let be $\Omega \subset \boldsymbol{R}^N$ a bounded open set with smooth boundary $\partial\Omega$, $\Gamma_0 \subset \partial\Omega$ a bounded open set in $\partial\Omega$ with smooth boundary, $\Gamma = \overline{\Gamma}_0$. Let be $a_{ij}(t, x)$ in $\mathscr{L}^\infty(\boldsymbol{R} \times \Omega)$ $(i, j = 1, ..., N)$ with

$$\sum_{ij=1}^{N} a_{ij}(t, x)\,\xi_i \xi_j^x \geqslant \alpha\,|\xi|^2 \qquad \text{a.e. in } \boldsymbol{R}\times\Omega\,, \tag{1.1}$$

$$t \to a_{ij}(t, \cdot) \qquad \text{almost periodic in } \mathscr{L}^\infty(\Omega)\,. \tag{1.2}$$

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Let be V the subspace of $H^1(\Omega)$ defined by

$$\{v \in H^1(\Omega);\ v|_\Gamma = 0\} \tag{1.3}$$

$H = \mathscr{L}^2(\Omega)$, V^* the dual of V for the duality $\langle\,,\rangle$, $V \hookrightarrow H \hookrightarrow V^*$. Let be $A(t)\colon V \to V^*$ defined by

$$\langle A(t)u, v\rangle = \sum_{ij=1}^{N} \int_\Omega a_{ij}(t,x) \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x)\,dx + \lambda \int_\Omega u(x)v(x)\,dx\,, \tag{1.4}$$

($\lambda \geqslant 0$ if $\Gamma = \partial\Omega$, $\lambda > 0$ if $\Gamma \neq \partial\Omega$) and $M\colon \mathscr{L}^\infty(\Omega) \to \mathscr{L}^\infty(\Omega)$ such that

$$M\varphi(x) = 1 + \operatorname*{Inf}_{x+\zeta\in\Omega,\,\zeta\geqslant 0} \varphi(x+\zeta)\,. \tag{1.5}$$

We indicate

$$K^\psi = \{v \in V,\ v(x) \leqslant \psi(x) \text{ a.e. in } \overline{\Omega}\}\,, \qquad \psi \in \mathscr{L}^2(\Omega) \text{ such that } K^\psi \neq \phi\,.$$

We consider the problem

$$\langle u'(t) + A(t)u(t) - f(t),\, v(t) - u(t)\rangle \geqslant 0 \text{ a.e.}\,,$$

$$v \in \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; V)\,, \qquad v(t) \in K^{\psi(t)} \text{ a.e.}\,, \tag{1.6}$$

$$u(t) \text{ almost periodic in } C(\overline{\Omega}),\ u \in H^1_{\text{loc}}(\boldsymbol{R}; V^*) \cap \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; V),\ u(t) \in K^{\psi(t)} \text{ a.e.},$$

where, $\forall v \in V$,

$$\langle f(t), v\rangle = \sum_{i=1}^{N} \int_\Omega f_i(t,x) \frac{\partial v}{\partial x_2}(x)\,dx + \int_\Omega f_0(t,x)v(x)\,dx\,,$$

with $t \to f_i(t,\cdot)$ almost periodic in $\mathscr{L}^p(\Omega)$, $p > N$ $(i = 0, 1, \ldots, N)$.

Theorem 1. *Let be $g(t) = \psi'(t) + A(t)\psi(t)$ almost periodic in $\mathscr{L}^\infty(\Omega)$; the problem* (1.6) *has a unique solution.*

The Th. 1 allow us to define a mild solution of (1.6) in the case $\psi(t,\cdot)$ almost periodic in $C(\overline{\Omega})$.

We indicate by $u = S_f\psi$ the solution to (1.6) in the case of Th. 1.

Theorem 2. *The operator S_f has a unique continuous extension S_f to the space of the $\underset{\Gamma}{C}(\overline{\Omega})$-almost periodic functions.*

Definition 1. *Let be $\psi(t,\cdot)$ $\underset{\Gamma}{C}(\overline{\Omega})$-almost periodic, $u = S_f\psi$ is the mild solution of* (1.6).

We consider now the problem

$$(1.6)' \quad \begin{aligned} &\langle u'(t) + A(t)u(t) - f(t), v(t) - u(t)\rangle \geq 0 \text{ a.e.},\\ &\forall v \in \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; V), \quad v(t) \in K^{Mu(t)} \text{ a.e.},\\ &u(t) \text{almost periodic in } C(\bar{\Omega}),\ u \in H^1_{\text{loc}}(R; V^*) \cap \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; V),\ u(t) \in K^{Mu(t)} \text{ a.e.} \end{aligned}$$

Definition 2. *The function* $u_0(t, x)$ *is a mild subsolution of* (1.6′) *iff.*

$$u_0 \leq S_f\, u_0 .$$

Theorem 3. *Let be* $M: C(\bar{\Omega}) \to C(\bar{\Omega})$, u_0 *a mild subsolution such that*

(I) $u_0'(t) + A(t)u_0(t)$ *is* $(W^{1,p'}_\Gamma)^*$*-almost periodic* $(1/p + 1/p' = 1,\ W^{1,p'}_\Gamma(\Omega) = \{v \in W^{1,p'}(\Omega),\ v|_\Gamma = 0\})$,

(II) $u_0'(t) + A(t)u_0(t) \leq f(t)$ *in* $(W^{1,p'}_\Gamma)^*$,

(III) $u_0(t, x) \geq -1 + \delta$ a.e. *in* $\boldsymbol{R} \times \Omega$, $\delta > 0$.

The problem (1.6)′ *has a unique solution.*

In the n. **2** we give a proof of Th. 1 and in the n. **3**, we show the Th. 2; the proof of the two results uses some classical methods in almost periodicity, a result of Charrier-Tronieniello [**4**] with C^α-regularity for parabolic equations.

In the n. **4** we give a proof of Th. 3, which uses an iterative method [**2**], [**9**] and an extension to our case of the estimate on convergence of iterates in $\mathscr{L}^\infty(\Omega)$ given in [**3**]$_4$ [**6**].

Remark 1. The Th. 2 has been given by T. Norando [**10**] in the case $A(t) = A$, $\Gamma = \partial\Omega$.

Remark 2. The condition $M: C(\bar{\Omega}) \to C(\bar{\Omega})$ holds in the case $\Gamma = \partial\Omega$; in the case $\Gamma \neq \partial\Omega$ is a condition on geometrical properties of Ω, which holds if $\bar{x}_i \to \Omega_{\bar{x}_i}$ ($\Omega_{\bar{x}_i}$ section of Ω with the hyperplane $x_i = \bar{x}_i$) is continuous in the Hausdorf topology or if Ω is strictly convex.

2. - Proof of Theorem 1

Let be $u_n(t)$ the solution to the problem

$$(2.1) \quad \begin{aligned} &\langle u_n'(t) + A(t)u_n(t) - f(t), v(t) - u_n(t)\rangle \geq 0 \text{ a.e. } t \in [-n, +\infty[,\\ &\forall v \in \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; V), \quad v(t) \in K^{\psi(t)} \text{ a.e. } t \in [-n, +\infty[,\\ &u_n \in H^1(-n, +\infty; V^*) \cap \mathscr{L}^2(-n, +\infty; V), \quad u_n(t) \in K^{\psi(t)} \text{ a.e.},\\ &t \in [-n, +\infty[. \end{aligned}$$

We indicate again by $u_n(t)$ the prolongate of $u_n(t)$ to $\boldsymbol{R}$ by 0.

By the same methods used in [**3**]$_1$ we have that, at more after an extraction of subsequence,

$$\lim_{n\to\infty}{}^{*}\, u_n(t) = u(t) \quad \text{in } \mathscr{L}^2_{\mathrm{loc}}(\boldsymbol{R};\, V)\,, \tag{2.2}$$

$$\lim_{n\to\infty} u_n(t) = u(t) \quad \text{in } \mathscr{L}^\infty_{\mathrm{loc}}(\boldsymbol{R};\, H)\,, \tag{2.3}$$

and from [**4**] we have

$$\|u_n' + A(\cdot)\,u_n\|_{\mathscr{L}^\infty(-n,+\infty;\, \mathscr{L}^\infty(\Omega))} \leqslant C\,. \tag{2.4}$$

From (2.2), (2.3), (2.4) we have easily that $u(t)$ is the solution to the problem

$$\begin{aligned} &\langle u'(t) + A(t)u(t) - f(t),\, v(t) - u(t)\rangle \geqslant 0 \ \text{a.e.}\,,\\ &\forall v \in \mathscr{L}^2_{\mathrm{loc}}(\boldsymbol{R};\, V)\,, \qquad v(t) \in K^{\Psi(t)} \ \text{a.e.}\,,\\ &u(t) \in H^1_{\mathrm{loc}}(\boldsymbol{R};\, V^*) \cap \mathscr{L}^2_{\mathrm{loc}}(\boldsymbol{R};\, V) \cap \mathscr{L}^\infty(\boldsymbol{R};\, H)\,, \qquad u(t) \in K^{\Psi(t)} \ \text{a.e.}\,. \end{aligned} \tag{2.5}$$

We can show as in [**3**]$_2$ that the solution to (2.5) is unique and from (2.4)

$$\|u' + A(\cdot)\,u\|_{\mathscr{L}^\infty(Q)} \leqslant C\,, \tag{2.6}$$

where $Q = \boldsymbol{R}\times\Omega$. From (2.6) and [**7**] we have easily $u \in \mathscr{L}^\infty\big(\boldsymbol{R};\, C^\alpha(\bar{\Omega})\big)$, $0<\alpha<1$. From (2.6), being $u \in \mathscr{L}^\infty\big(\boldsymbol{R};\, C^\alpha(\bar{\Omega})\big)$, we can easily show by standard methods in almost periodicity that u is almost periodic in $C(\bar{\Omega})$.

3. - Proof of Theorem 2

Lemma 1. *Let be* $\psi \in W^{1,\infty}_T\big(\boldsymbol{R};\, \mathscr{L}^p(\boldsymbol{R}^N)\big) \cap \mathscr{L}^\infty\big(\boldsymbol{R}^N;\, W^{1,p}(\boldsymbol{R}^N)\big)$ *and*

$$n^{-1}\big(\psi_n'(t) + A(t)\,\psi_n(t)\big) + \psi_n(t) = \psi(t) + n^{-1}\psi'(t)\,. \tag{3.1}$$

The problem (3.1) *has a unique solution* $\psi_n \in \mathscr{L}^\infty(\boldsymbol{R}^{N+1})$ *and*

$$\|\psi' + A(\cdot)\,\psi\|_{\mathscr{L}^\infty(\boldsymbol{R}^{N+1})} \leqslant K n^{\frac12}\,, \tag{3.2}$$

$$\|\psi_n - \psi\|_{\mathscr{L}^\infty(\boldsymbol{R}^{N+1})} \leqslant K n^{-\frac12}\,. \tag{3.3}$$

If ψ is almost periodic in $W^{1,p}(\mathbf{R}^N)$ and ψ' is almost periodic in $\mathscr{L}^p(\mathbf{R}^N)$, ψ_n is almost periodic in $C(\bar{\Omega})$.

The proof of the Lemma 1 uses a method given by E. De Giorgi, S. Spagnolo [**5**], in the elliptic case.

Let be $w_n = n(\psi_n - \psi)$ we have

$$n^{-1}\big(w_n' + A(t)w_n(t)\big) + w_n(t) = A(t)\psi(t)\,.$$

Using the transformation $\tau = n^{-1}t$ and (6.11) of [7] (p. 105), we have (3.3) and from (3.1), (3.3) we have (3.2). If $\psi(t)$ is almost periodic in $W^{1,\infty}(\mathbf{R}^N)$ with $\psi'(t)$ almost periodic in $\mathscr{L}^p(\mathbf{R}^N)$ from the linearity of (3.1) we have that $\psi_n(t)$ is almost periodic in $C(\mathbf{R}^N)\cap\mathscr{L}^\infty(\mathbf{R}^N)$.

As in [**11**], [**10**] we have

Lemma 2. *Let be $\psi_i\in\mathscr{L}^\infty(Q)$, $\psi_n'(t) + A(t)\psi_i(t)\in\mathscr{L}^\infty(Q)$ $(i = 1, 2)$ $Q = \mathbf{R}\times\Omega$, with $g_i(t) = \psi_n'(t) + A(t)\psi_i(t)$ almost periodic in $\mathscr{L}^\infty(\Omega)$; we have*

$$\|\tilde{S}_f\psi_1 - \tilde{S}_f\psi_2\|_{\mathscr{L}^\infty(\Omega)} \leqslant \|\psi_1 - \psi_2\|_{\mathscr{L}^\infty(\Omega)}\,.$$

From the Lemma 1 we have that the set N of functions $\{\psi\in\mathscr{L}^\infty(Q);\ \psi'(t) + A(t)\psi(t)$ almost periodic in $\mathscr{L}^\infty(\Omega)\}$ is dense in the set of $C(\bar{\Omega})$-almost periodic functions with $\psi|_\Gamma = 0$.

From the Lemma 2 we have that if $\{\psi_n\}$ N and $\lim_{n\to\infty}\psi_n = \psi$ in $\mathscr{L}^\infty(Q)$, we have

$$\lim_{n\to\infty}\tilde{S}_f\psi_n = \chi \quad \text{in } \mathscr{L}^\infty(Q)\,,$$

where χ depends only on ψ but not on the sequence $\{\psi_n\}$. We can define $\chi = S_f\psi$ and it is easy to verify that S_f is the unique continuous extension of $\tilde{S}_f$ to the space of $\underset{\Gamma}{C}(\bar{\Omega})$-almost periodic functions the result fullow easily.

4. - Proof of Theorem 3

Lemma 1. *The map $f\to S_f\psi$ is increasing.*

It is enough to show the lemma in the case $\psi\in N$.

Let be $f_1\geqslant f_2$ in $(w_\Gamma^{1,p'})^*$; we choose in (1.6) $v = (u_1 + u_2)/2 + (u_2 - u_1)^-/2$ $(u_1 = S_{f_1}\psi,\ u_2 = S_{f_2}\psi)$. We have

$$\langle w'(t), w^+(t)\rangle + \langle A(t)w(t), w^+(t)\rangle \leqslant 0\,,$$

where $w = u_2 - u_1$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^+(t)\|^2_{\mathscr{L}^2} + \alpha\|w^+(t)\|^2_V \leqslant 0\,. \tag{4.1}$$

By the same methods used in $[\mathbf{3}]_2$, we have from (4.1) $w^+ = 0$.

Lemma 2. *The map* $\psi \to S_f \psi$ *is increasing.*

It is enough to show the Lemma 2 for $\psi \in N$.

Let be $\psi_1 \geqslant \psi_2$, $\psi_1, \psi_2 \in N$; we choose in (1.6) $v = (u_1 + u_2)/2 + (u_2 - u_1)^-/2$ $(u_1 = S_f \psi_1,\ u_2 = S_f \psi_2)$. We have

$$\langle w'(t), w^+(t)\rangle + \langle A(t) w(t), w^+(t)\rangle \leqslant 0 ,$$

where $w = u_2 - u_1$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^+(t)\|^2_{\mathscr{L}^2} + \alpha\|w^+(t)\|^2_{V} \leqslant 0 . \tag{4.2}$$

By the same methods used in $[\mathbf{3}]_2$ we have from (4.2) $w^+ = 0$.

Lemma 3. *Let be* $T(\psi, f, g)$ $(g = \text{const})$ *the mild solution of the problem*

$$\begin{aligned}
&\langle u'(t) + A(t)u(t) - f(t), v(t) - u(t)\rangle \geqslant 0 \text{ a.e.},\\
&\forall v \in \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; H^2(\Omega)) , \quad v(t) \in K^{\Psi(t)} \text{ a.e.}, \quad v(t,\cdot)|_\Gamma = g ,\\
&u(t),\ C(\bar{\Omega})\text{-almost periodic in } C(\bar{\Omega});\ u \in H^1_{\text{loc}}(\boldsymbol{R}; V^*) \cap \mathscr{L}^2_{\text{loc}}(\boldsymbol{R}; V),\\
&u(t) \in K^{\Psi(t)} \text{ a.e.},\ u(t,\cdot)|_\Gamma = g .
\end{aligned} \tag{4.3}$$

The map T *is increasing in* f, ψ *and* g.

The result is a consequence of two predicing lemmas.

Remark 1. We can verify that the lemmas 1, 2, 3 holds again for $M\colon C(\bar{\Omega}) \to C(\bar{\Omega})$ increasing.

Lemma 4. *Let be* $u^n = S_f(Mu^{n-1})$ *and* u^0 *given by*

$$(u^0)'(t) + A(t)u^0(t) = f(t) , \qquad u^0(t) \in V \text{ a.e.}.$$

The sequence $\{u^n\}$ *converges in* $C(\bar{\Omega})$ *to a fixed point* $\bar{u}$ *of* $\psi \to S_f(M\psi)$ *and*

$$\|u^n - \bar{u}\|_{\mathscr{L}^\infty(a)} \leqslant K\theta^n ,$$

where K, $0 < \theta < 1$ *depends only on* u^0, u_0, δ.

The method used in the proof is analogous to the method used in $[\mathbf{3}]^4$, $[\mathbf{5}]$.

From the hypothesis (I), (II), (III) we can suppose $f \geqslant 0$, $u_0 = 0$

$$M_u \to M'u = 1 + u^0(t, x) + \operatorname*{Inf}_{x+\xi \in \Omega, \xi \geqslant 0} (u - u^0)(t, x + \xi) .$$

We observe that $M': C(\bar{\Omega}) \to C(\bar{\Omega})$ is increasing, $\forall t \in \mathbf{R}$. Being $f \to S_f(\psi)$ increasing, the sequence $\{u^n\}$ is decreasing and non negative; then

$$\lim_{n\to\infty}{}^{*}\ u^n = \bar{u} \quad \text{in } V\,, \qquad \bar{u} = \bigwedge_{n=1}^{\infty} u^n\,.$$

To show the result it is enough to show

$$u^n + R \leqslant \frac{\theta^{-n} + C}{\theta^{-n}}(u^p + R) \qquad \forall p\,, \tag{4.4}$$

where R, C, $0 < \theta < 1$ don't depend on n, p. We use a proof by induction.

For $n = 0$, if we choose $CR \geqslant \sup\limits_{(t,x)\in Q} u^0(t, x) = D$, (4.3) holds.

We suppose now that (4.4) holds for $n - 1$ and we show that (4.4) holds for n.

Let $w^n = u^n + R$ we suppose

$$w^{n-1} \leqslant \frac{\theta^{-n+1} + C}{\theta^{-n+1}} w^p \qquad \forall p\,, \tag{4.5}$$

and we show

$$w^n \leqslant \frac{\theta^{-n} + C}{\theta^{-n}} w^p \qquad \forall p\,. \tag{4.4'}$$

We have

$$w^n = T(M' w^{n-1}, f + \lambda R, R)\,,$$

then

$$\frac{\theta^{-n}}{\theta^{-n} + C} w^n = T\Big(\frac{\theta^{-n}}{\theta^{-n} + C} M' w^{n-1}, \frac{\theta^{-n}}{\theta^{-n} + C}(f - \lambda R), \frac{\theta^{-n}}{\theta^{-n} + C} R\Big) \tag{4.6}$$

$$\leqslant T\Big(\frac{\theta^{-n}}{\theta^{-n} + C} M' w^{n-1}, f - \lambda R, R\Big)\,.$$

As in [9] (p. 168) we have

$$M'(\alpha\, w^{n-1}) \geqslant \frac{\theta^{-n}}{\theta^{-n} + C} M' w^{n-1}\,, \tag{4.7}$$

if

$$\frac{1-\theta^{-n}/(\theta^{-n}+C)}{\theta^{-n}/(\theta^{-n}+C)-\alpha}\geqslant\delta^{-1}(D+R)=\bar{D},$$

then we can choose

$$\alpha=\operatorname{Max}\left(\frac{\bar{D}\theta^{-n}-C}{\bar{D}(\theta^{-n}+C)},0\right). \tag{4.8}$$

From (4.6) (4.7) we have

$$\frac{\theta^{-n}}{\theta^{-n}+C}w^n\leqslant T\left(M'(\alpha w^{n-1}),f,R\right)$$
$$\leqslant T\left(M'\left(\alpha\frac{\theta^{-n+1}}{\theta^{-n+1}+C}w^p\right),f,R\right)\qquad\forall p.$$

We have now the result if

$$\alpha\frac{\theta^{-n+1}}{\theta^{-n+1}+C}\leqslant 1. \tag{4.9}$$

Choosing $\theta=\bar{D}/(\bar{D}+1)$, we have (4.9) and the result.

From the Lemma 4 we have that $\bar{u}$, which is $C(\bar{\Omega})$-almost periodic is a fixed point of $\psi\to S_f(M\psi)$, then it is a solution to the problema (1.6)$'$.

To show the uniqueness of a solution to (1.6)$'$ we use a method given by Th. Laestch, [**8**].

We observe: u mild solution to (1.6)$'$ $\Leftrightarrow$ u fixed point of $\psi\to S_f(M\psi)$. It is easy to verify that $\bar{u}$ is the maximum fixed point of $\psi\to S_f(M\psi)$ in $\{v\,|\,v\geqslant u_0\}$.

We use the transformation $u\to w=u-u_0$, $M\to M'$.

We have now $f\geqslant 0$, $w_0=0$.

We observe that M' is such that $\forall\varphi\geqslant 0$, $C(\bar{\Omega})$-almost periodic and $0\leqslant\bar{\alpha}<1$ there is $\bar{\alpha}<\beta<1$ such that $M'(\alpha\varphi)\geqslant\beta M'(\varphi)$.

Let be now $\bar{w}$ the maximum positif fixed point for $\psi\to S_f(M'\psi)$.

Let be $w\geqslant 0$ a different solution of (1.6)$'$ and $\bar{\alpha}$ the greatest real such that $\bar{\alpha}\bar{w}\leqslant w$, $\bar{\alpha}\geqslant 0$.

If $\alpha=1$ we have $w=\bar{w}$, then $\bar{\alpha}<1$.

There is $\bar{\alpha}<\beta<1$ such that $M'(\alpha\bar{w})\geqslant\beta M'(\bar{w})$, then being M', $\psi\to S_f(\psi)$ increasing $\beta\bar{w}\leqslant w$.

We have a contradiction and the result is shown.

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Riassunto

Si dà un risultato di esistenza ed unicità per la soluzione $C(\Omega)$ quasi periodica di certe disequazioni variazionali.

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