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MARCO MODUGNO (*)

On the structure of classical kinematics.

Absolute kinematics (**)

A GIORGIO SESTINI per il suo 70° compleanno

In this paper we study the general event framework constituted by the event space, its partition into the simultaneity spaces, which generate the time, and the spatial metric.

We analyse some remarkable spaces and maps connected with the previous ones. Finally we study the one-body absolute motion, velocity and acceleration. All these elements are considered regardless of any frame of reference.

1. - The event space

First we introduce the general framework for classical mechanics.
Event space, simultaneity, spatial metric, future orientation, time.

1.1. – Basic assumptions on primitive elements of our theory are given by the following definition, which constitutes the framework of classical mechanics.

Definition. The *classical event framework* is a 4-plet

$$\theta \equiv \{E, \overline{S}, \check{G}, 0\}$$

(*) Indirizzo: Via G. Prati 20, 50124 Firenze, Italy.

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where

$E \equiv \{\boldsymbol{E}, \bar{\boldsymbol{E}}, \sigma\}$ is an affine space, with dimension 4;

$\bar{\boldsymbol{S}} \hookrightarrow \bar{\boldsymbol{E}}$ is a subspace of $\bar{\boldsymbol{E}}$, with dimension 3;

$\check{\boldsymbol{G}}$ is a conformal euclidean metric on $\bar{\boldsymbol{S}}$;

0 is an orientation on the quotient space $\bar{\boldsymbol{E}}/\bar{\boldsymbol{S}}$.

$\boldsymbol{E}$ is the *event space*; $\bar{\boldsymbol{E}}$ is the *event interval space*;

$\bar{\boldsymbol{S}}$ is the *simultaneous interval space* or the *spatial interval space*;

$\check{\boldsymbol{G}}$ is the *spatial conformal metric*;

0 is the *future orientation*;

-0 is the *past orientation*.

Henceforth we assume a classical event framework θ to be given.

1.2. – The previous definition contains implicitly the notion of absolute time, which we are now giving explicitly.

Definition. The *time space* is the quotient space

$$\boldsymbol{T} \equiv \boldsymbol{E}/\bar{\boldsymbol{S}} .$$

The *time vector space* is the quotient space

$$\bar{\boldsymbol{T}} \equiv \bar{\boldsymbol{E}}/\bar{\boldsymbol{S}} .$$

The *time projection* is the quotient map

$$t\colon \boldsymbol{E} \to \boldsymbol{T} .$$

The *space at the time* $\tau \in \boldsymbol{T}$ is the subspace

$$\boldsymbol{S}_\tau \equiv t^{-1}(\tau) \hookrightarrow \boldsymbol{E} .$$

The *time bundle* is the 3-plet

$$\eta \equiv (\boldsymbol{E}, t, \boldsymbol{T}) .$$

Hence, each equivalence class is of the type

$$\boldsymbol{T} \in \tau \equiv [e] \equiv e + \overline{\boldsymbol{S}} \equiv \boldsymbol{S}_\tau \hookrightarrow \boldsymbol{E}\,, \qquad \text{having} \quad t(e) \equiv \tau\,.$$

Thus τ and $\boldsymbol{S}_\tau$ coincide, but τ is viewed as a point of $\boldsymbol{T}$ and $\boldsymbol{S}_\tau$ as a subset of $\boldsymbol{E}$.

Moreover we will denote by j the injective map

$$j \equiv (t, id_{\boldsymbol{E}}) \colon \boldsymbol{E} \hookrightarrow \boldsymbol{T} \times \boldsymbol{E}\,.$$

1.3. – We get immediate properties for the previous spaces.

Proposition.

(a) $(\boldsymbol{T}, \overline{\boldsymbol{T}})$ results naturally into an affine 1-dimensional oriented space.

(b) t is an affine surjective map. We get $\overline{\boldsymbol{S}} = (Dt)^{-1}(0)$.

(c) For each $\tau \in \boldsymbol{T}$, $(\boldsymbol{S}_\tau, \overline{\boldsymbol{S}}, \sigma)$ is an affine 3-dimensional subspace of $\boldsymbol{E}$; hence $\{\boldsymbol{S}_\tau\}_{\tau \in \boldsymbol{T}}$ is a family of parallel, (not canonically) isomorphic affine subspaces of $\boldsymbol{E}$ and we have $\boldsymbol{E} = \bigcup_{\tau \in \boldsymbol{T}} \boldsymbol{S}_\tau$.

(d) η is an affine, (not canonically) trivial bundle.

1.4. – We have absolute time component of an event interval.

Definition. The *time component* of the vector $u \in \overline{\boldsymbol{E}}$ is $u^0 \equiv \langle Dt, u\rangle \in \overline{\boldsymbol{T}}$. u is *future oriented* or *past oriented*, according as $u^0 \in \overline{\boldsymbol{T}}^+$ or $u^0 \in \overline{\boldsymbol{T}}^-$. Moreover u is spatial if and only if $u^0 = 0$.

1.5. – Thus, the sequence

$$0 \to \overline{\boldsymbol{S}} \hookrightarrow \overline{\boldsymbol{E}} \to \overline{\boldsymbol{T}} \to 0$$

is exact, but we have not a canonical splitting of $\overline{\boldsymbol{E}}$, as we have not a canonical projection $\overline{\boldsymbol{E}} \to \overline{\boldsymbol{S}}$, or a canonical inclusion $\overline{\boldsymbol{T}} \hookrightarrow \overline{\boldsymbol{E}}$. However, each vector $v \in \overline{\boldsymbol{E}}$, such that $\langle Dt, v\rangle \neq 0$, determines a splitting of $\overline{\boldsymbol{E}}$.

Namely we get the inclusion

$$\overline{\boldsymbol{T}} \hookrightarrow \overline{\boldsymbol{E}}\,, \qquad \text{given by} \quad \lambda \mapsto \frac{\lambda}{v^0}\, v\,,$$

and the projection

$$p_v^{\perp}\colon \overline{E}\to\overline{S}\,,\qquad \text{given by}\qquad u\mapsto u-\frac{u^0}{v^0}v\,,$$

which determine the decomposion in the direct sum

$$\overline{E}=\overline{T}\oplus\overline{S}\qquad \text{given by}\qquad u=\frac{u^0}{v^0}v+(u-\frac{u^0}{v^0}v)\equiv p_v^{\parallel}(u)+p_v^{\perp}(u)\,.$$

1.6. – According to the bundle structure of $\boldsymbol{E}$ on $\boldsymbol{T}$, we can define the vertical derivative of maps, i.e. the derivative along the fibers. Generally we will denote by «$\smallsmile$» the vertical quantities connected with η.

Definition. Let $\boldsymbol{F}$ be an affine space and let $f\colon \boldsymbol{E}\to\boldsymbol{F}$ be a C^∞ map. The *vertical derivative* of f is the map

$$\check{D}f\equiv Df_{|\overline{S}}\colon \boldsymbol{E}\to\overline{\boldsymbol{S}}^*\otimes\overline{\boldsymbol{F}}\,.$$

1.7. – *Poincaré's and Galilei's maps.* A Poincaré's map is a map $\boldsymbol{E}\to\boldsymbol{E}$ which preserves the structure of θ and the associated Galilei's map is its derivative.

Definition. A *Poincaré's map* is an affine map

$$G\colon \boldsymbol{E}\to\boldsymbol{E}\,,$$

such that

(a) $DG(\overline{\boldsymbol{S}})=\overline{\boldsymbol{S}}$,

(b) $DG\in U(\overline{\boldsymbol{S}})$,

(c) if $G^0\colon \boldsymbol{T}\to\boldsymbol{T}$ is the induced map on the quotient space $\boldsymbol{T}\equiv\boldsymbol{E}/\overline{\boldsymbol{S}}$, then $DG^0=id_{\overline{T}}$.

$DG\colon \overline{\boldsymbol{E}}\to\overline{\boldsymbol{E}}$ is the *Galilei's map* associated with G.

G is *special* if it preserves the orientations of $\overline{\boldsymbol{E}}$ and $\overline{\boldsymbol{S}}$ (hence of $\overline{\boldsymbol{T}}$).

1.8. – Proposition. Each Poincaré's map G is bijective.

Proof. It follows from $DG\in U(\overline{\boldsymbol{S}})$, $DG^0=id_{\overline{T}}$.

1.9. – *Space and time measure unity.* We have assumed a 1-parameter family $\check{\boldsymbol{G}}$ of euclidean metrics on $\bar{\boldsymbol{S}}$. A 1-parameter family $\boldsymbol{G}^0$ of euclidean metrics on $\bar{\boldsymbol{T}}$ is given a priori, for $\dim \bar{\boldsymbol{T}} = 1$.

An arbitrary choice of one among these makes important simplifications in the following.

Definition. A *spatial measure unity* is a metric $\check{g} \in \check{\boldsymbol{G}}$. A *time measure unity* is a metric $g^0 \in \boldsymbol{G}^0$.

The choice of a spatial measure unity $\check{g}$ is equivalent to the choice of the sphere (in the family determined by $\check{\boldsymbol{G}}$) of $\bar{\boldsymbol{S}}$, with radius 1 as measured by $\check{g}$.

The choice of a time measure unity g^0 is equivalent to the choice of the vector

$$\lambda^0 \in \boldsymbol{T}^+ \quad \text{such that} \qquad g^0(\lambda^0, \lambda^0) = 1 .$$

Then g^0 determines the isomorphism

$$\bar{\boldsymbol{T}} \to \boldsymbol{R} \quad \text{given by} \quad \lambda \mapsto \frac{\lambda}{\lambda^0} .$$

Henceforth we assume a spatial and a time measure unity to be given. Hence we get the identification

$$\bar{\boldsymbol{T}} \cong \boldsymbol{R}$$

and the consequent identifications

$$L(\bar{\boldsymbol{T}}, \bar{\boldsymbol{E}}) \cong \bar{\boldsymbol{E}} , \qquad L(\boldsymbol{T}, \bar{\boldsymbol{S}}) \cong \bar{\boldsymbol{S}} , \qquad L(\bar{\boldsymbol{E}}, \bar{\boldsymbol{T}}) \cong \bar{\boldsymbol{E}}^* , \qquad L(\bar{\boldsymbol{S}}, \bar{\boldsymbol{T}}) \cong \bar{\boldsymbol{S}}^* , \qquad \dots .$$

In this way, the map $Dt \in L(\bar{\boldsymbol{E}}, \bar{\boldsymbol{T}})$ is identified with the form

$$\underline{t} \cong Dt \in \bar{\boldsymbol{E}}^* .$$

1.10. – Besides the subspace $\bar{\boldsymbol{S}} \hookrightarrow \bar{\boldsymbol{E}}$, which results into $\bar{\boldsymbol{S}} = \underline{t}^{-1}(0)$, an interesting role will be played by the subspace of normalized vectors $\underline{t}^{-1}(1)$.

Definition. The *free velocity space* is

$$\boldsymbol{U} \equiv \underline{t}^{-1}(1) \hookrightarrow \bar{\boldsymbol{E}} .$$

1.11. – Proposition. $(\boldsymbol{U}, \overline{\boldsymbol{S}})$ results naturally into an affine (not vector) 3-dimensional subspace of $\overline{\boldsymbol{E}}$.

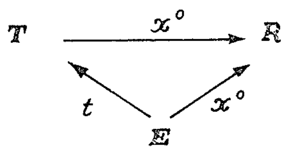
Of course $\boldsymbol{U}$ and $\overline{\boldsymbol{S}}$ are isomorphic as affine spaces, but we have not a canonical affine isomorphism between $\boldsymbol{U}$ and $\overline{\boldsymbol{S}}$.

1.12. – *Special charts.* In calculations can be useful a numerical representation of $\boldsymbol{E}$, which takes into account its time structure. For simplicity of notations, we consider only diffeomorphism $\boldsymbol{E} \to \boldsymbol{R}^4$, leaving to the reader the obvious generalization to local charts, our considerations being essentially local.

Definition. A *special chart* is a C^∞ chart

$$x \equiv \{x^0, x^i\} : \boldsymbol{E} \to \boldsymbol{R} \times \boldsymbol{R}^3 ,$$

such that x^0 factorizes as follows



where $\underset{\smile}{x}^0 : \boldsymbol{T} \to \boldsymbol{R}$ is a normal oriented cartesian map. Naturally $\underset{\smile}{x}^0$ (hence x^0) is determined up an initial time.

We make the usual convention

$$\alpha, \beta, \lambda, \mu, \ldots = 0, 1, 2, 3 \qquad \text{and} \qquad i, j, h, k, \ldots = 1, 2, 3 .$$

We assume in the following a special chart x to be given.

1.13. – Let us give the coordinate expression of some important quantities.

Proposition. We have

$$Dx^0 = \underline{t} , \qquad \delta x_i : \boldsymbol{E} \to \overline{\boldsymbol{S}} ;$$

if $u \in \overline{\boldsymbol{E}}$, then $u = u^0 \, \delta x_0 + u^i \, \delta x_i$, where $u^0 \equiv \langle \underline{t}, u \rangle$;

$$\check{g} = g_{ij} \check{D}x^i \otimes \check{D}x^j ;$$

$$\Gamma^0_{\alpha\beta} \equiv D\delta x_\alpha(\delta x_\beta, Dx^0) = - D^2x^0(\delta x_\alpha, \delta x_\beta) = 0 ,$$

$$\Gamma^k_{ij} \equiv D\delta x_i(\delta x_j, Dx^k) = - D^2x^k(\delta x_i, \delta x_j) = \tfrac{1}{2} g^{kh}(\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}) ,$$

$$\Gamma_{i,0j} + \Gamma_{j,0i} = \partial_0 g_{ij}, \text{ where } \Gamma_{h,\alpha\beta} \equiv g_{hi}\Gamma^i{}_{\alpha\beta} .$$

Moreover

$$\Gamma^k_{0j} \equiv D\delta x_0(\delta x_j, Dx^k) = - D^2x^k(\delta x_0, \delta x_j)$$

and

$$\Gamma^k_{00} \equiv D\delta x_0(\delta x_0, Dx^k) = - D^2x^k(\delta x_0, \delta x_0)$$

can be different from zero, if δx_0 is not constant.

Notice that $Dx^0 = \underline{t}$ is fixed a priori and that the unique conditions imposed a priori on δx_α are $\langle \underline{t}, \delta x_0 \rangle = 1$, $\langle \underline{t}, \delta x_i \rangle = 0$.

1.14. – *Physical description.* The event space $\boldsymbol{E}$ represents the set of all the possible events considered from the point of view of their mutual space-time collocation and without reference to any particular frame of reference. This space $\boldsymbol{E}$ must be viewed exactly in the same sense as the event space of Special and General Theory of Relativity.

The event space $\boldsymbol{E}$ is the disjoint union of a family $\{\boldsymbol{S}_\tau\}_{\tau \in \boldsymbol{T}}$ of three dimensional affine euclidean, mutually diffeomorphic, spaces. This partition represents the equivalence relation of absolute simultaneity among events. The structure of each space $\boldsymbol{S}_\tau$ permits all the physical operations considered in the classical time-independent Euclidean Geometry, as stright lines, parallelism, intervals, sum of intervals, by the parallelogram rule, circles, etc. We have not selected a priory a spatial measure unity, for it is not physically significant: by means of rigid rods we can only find ratios between lengths in all directions and the choice of a particular interval of a rigid rod is a useful but not necessary convenction.

The simultaneity spaces $\boldsymbol{S}_\tau$ are mutually, but not canonically, isomorphic, for a particular family of bijections among these leads to a determination of positions, i.e. to a frame of reference, which we have excluded in the general context. Notice that in $\boldsymbol{S}_\tau$ we have not privileged points or axes.

The required four dimensional affine structure of $\boldsymbol{E}$ leads to the affine structures of the subspaces $\boldsymbol{S}_\tau$ and to the one dimensional affine structure of the set $\boldsymbol{T}$, whose points are the equivalence classes $\boldsymbol{S}_\tau$. This space represents the classical absolute time. Its affine structure admits the time intervals, independent of an initial time, and their sum. The one dimensional affine structure of $\boldsymbol{T}$ leads also to the measure of time intervals with respect to an arbitrary chosen unity. Hence the affine structure of $\boldsymbol{E}$ contains implicitly the idea of « good clocks ». The dimension one describes also the total ordinability of times and the assumed orientation describes the future orientation. Notice that in $\boldsymbol{T}$ we have not a privileged initial time.

2. - Further spaces and maps

Now we introduce some further notions concerning applied vector spaces and maps.

2.1. – *Vertical and unitary spaces.* We introduce the spaces of applied vectors relative to $\overline{\boldsymbol{S}}$, and $\boldsymbol{U}$.

Definition. The *vertical space, with respect to* $(\boldsymbol{E}, t, \boldsymbol{T})$, or the *phase space*, or the *acceleration space*, is

$$\boldsymbol{A} \equiv \check{T}\boldsymbol{E} \equiv \ker Tt = \boldsymbol{E}\times\overline{\boldsymbol{S}} \hookrightarrow T\boldsymbol{E}\,.$$

The *horizontal space with respect to* $(\boldsymbol{E}, t, \boldsymbol{T})$ is

$$\mathring{T}\boldsymbol{E} \equiv T\boldsymbol{E}_{/\check{T}\boldsymbol{E}} = \boldsymbol{E}\times\overline{\boldsymbol{T}}\,.$$

The *unitary space*, or the *velocity space*, is

$$\boldsymbol{V} \equiv \overset{|}{T}\boldsymbol{E} \equiv (Tt)^{-1}(\boldsymbol{T}\times 1) = \boldsymbol{E}\times\boldsymbol{U} \hookrightarrow \boldsymbol{E}\,.$$

2.2. – Let us remember that $T\boldsymbol{E}$ has two bundle structures, namely $(T\boldsymbol{E}, Tt, T\boldsymbol{T})$ and $(T\boldsymbol{E}, \pi_{\boldsymbol{E}}, \boldsymbol{E})$.

Proposition.

(a) $\check{T}\boldsymbol{E}$ is the submanifold of $T\boldsymbol{E}$ characterized by $\dot{x}^0 = 0$. $\overset{|}{T}\boldsymbol{E}$ is the submanifold of $T\boldsymbol{E}$ characterized by $\dot{x}^0 = 1$.

(b) $\check{T}\boldsymbol{E}$ and $\overset{|}{T}\boldsymbol{E}$ have two natural bundle structures, namely $(\check{T}\boldsymbol{E}, \check{t}, T)$ and $(\check{T}\boldsymbol{E}, \check{\pi}_{\boldsymbol{E}}, \boldsymbol{E})$, $(\overset{|}{T}\boldsymbol{E}, \overset{|}{t}, \boldsymbol{T})$ and $(\overset{|}{T}\boldsymbol{E}, \overset{|}{\pi}_{\boldsymbol{E}}, \boldsymbol{E})$.

(c) The sequence $0 \to \check{T}\boldsymbol{E} \hookrightarrow T\boldsymbol{E} \to \mathring{T}\boldsymbol{E} \to 0$ is exact.

We have not a canonical splitting of TE, as we have not a canonical projection $\check{T}E \to TE$, or a canonical inclusion $\mathring{T}E \hookrightarrow TE$.

In the same way we have not a canonical isomorphism $\check{T}E \to \overset{|}{T}E$.

2.3. – We can extend the vertical derivative in terms of applied vectors.

Definition. Let F be a C^∞ manifold and $f: E \to F$ a C^∞ map. The *vertical tangent map* of f, *with respect to* (E, t, T), is the map

$$\check{T}f \equiv Tf_{|\check{T}E} \quad \check{T}E \to TF .$$

2.4. – We can view the metric as a function on $\check{T}E$, which will become the kinetic energy in dynamics.

Definition. The *metric function* is the function

$$\check{g} : \check{T}E \to R , \qquad \text{given by} \qquad (e, u) \mapsto \tfrac{1}{2} u^2 .$$

2.5. – Proposition. We have $\check{g} = \frac{1}{2} \breve{g}_{ij}\, \dot{x}^i \dot{x}^j$.

2.6. – *Second order spaces, affine connection and canonical projection.* We consider now the second order tangent spaces.

Definition. The *vertical space, with respect to* $(\check{T}E, \check{t}, T)$, is

$$\check{T}^2 E \equiv \ker T\check{t} = E \times \bar{S} \times \bar{S} \times \bar{S} \hookrightarrow T^2 E .$$

The *vertical space, with respect to* $(\check{T}E, \check{t}, T)$ *and* $(\check{T}E, \check{\pi}_E, E)$, is

$$\nu\check{T}^2 E \equiv \ker T\check{t} \cap \ker T\pi_E = E \times \check{S} \times 0 \times \check{S} \hookrightarrow T^2 E .$$

The *biunitary space*, or *bivelocity space*, is

$$V^2 \equiv \overset{|}{T}{}^2 E \equiv ST\overset{|}{T}E \equiv E \times U \times \bar{S} \xrightarrow{\text{diagonal}} E \times U \times U \times \bar{S} \hookrightarrow T^2 E .$$

The *vertical biunitary space, with respect to* $(\overset{|}{T}E, \overset{|}{\pi}_E, E)$, is

$$\nu V^2 \equiv \nu\overset{|}{T}{}^2 E \equiv \nu T\overset{|}{T}E = E \times U \times 0 \times \bar{S} \hookrightarrow T^2 E .$$

2.7. – Proposition.

$\check{T}^2\boldsymbol{E}$ is the submanifold of $T^2\boldsymbol{E}$ characterized by

$$\overset{\smile}{\dot{x}}{}^0 = \dot{\overset{\smile}{x}}{}^0 = \ddot{x}^0 = 0\ .$$

$\nu\check{T}^2\boldsymbol{E}$ is the submanifold of $T^2\boldsymbol{E}$ characterized by

$$\overset{\smile}{\dot{x}}{}^0 = \dot{\overset{\smile}{x}}{}^\alpha = \ddot{x}^0 = 0\ .$$

$\overset{|}{T}{}^2\boldsymbol{E}$ is the submanifold of $T^2\boldsymbol{E}$ characterized by

$$\overset{\smile}{\dot{x}}{}^0 = \dot{\overset{\smile}{x}}{}^0 = 1\ , \qquad \overset{\smile}{\dot{x}}{}^i = \dot{\overset{\smile}{x}}{}^i\ , \qquad \ddot{x}^0 = 0\ .$$

$\nu\overset{|}{T}{}^2\boldsymbol{E}$ is the submanifold of $T^2\boldsymbol{E}$ characterized by

$$\overset{\smile}{\dot{x}}{}^0 = 1, \qquad \dot{\overset{\smile}{x}}{}^\alpha = 0\ , \qquad \ddot{x}^0 = 0\ .$$

2.8. – Let us consider some important canonical maps, which are used to define the covariant derivatives.

Definition. (a) The *affine connection map*

$$\Gamma\colon\ T^2\boldsymbol{E} \to \nu T^2\boldsymbol{E}\ , \qquad \text{given by} \qquad (e, u, v, w) \mapsto (e, u, o, w)\ ,$$

induces naturally the maps

$$\check{\Gamma}\colon\ \check{T}^2\boldsymbol{E} \to \nu\check{T}^2\boldsymbol{E} \qquad \text{and} \qquad \overset{|}{\Gamma}\colon\ \overset{|}{T}{}^2\boldsymbol{E} \to \nu\overset{|}{T}{}^2\boldsymbol{E}\ .$$

(b) The *canonical projection* (which is an isomorphism on fibers)

$$\coprod\colon\ \nu T^2\ \boldsymbol{E} \to T\boldsymbol{E}\ , \qquad \text{given by} \qquad (e, u, o, w) \mapsto (e, w)\ ,$$

induces naturally the maps

$$\check{\coprod}\colon\ \nu\check{T}^2\boldsymbol{E} \to \check{T}\boldsymbol{E}\ , \qquad \text{and} \qquad \overset{|}{\coprod}\colon\ \nu\overset{|}{T}{}^2\boldsymbol{E} \to \check{T}\boldsymbol{E}\ .$$

2.9. – Proposition. We have

$$\overset{\smile}{\overset{\smile}{x}}{}^\alpha\circ\Gamma = \overset{\smile}{\overset{\smile}{x}}{}^\alpha\ , \qquad \overset{\smile}{\dot{x}}{}^\alpha\circ\Gamma = \overset{\smile}{\dot{x}}{}^\alpha\ , \qquad \dot{\overset{\smile}{x}}{}^\alpha\circ\Gamma = 0\ , \qquad \ddot{x}^0\circ\Gamma = \ddot{x}^0;\ \ddot{x}^k\circ\Gamma = \ddot{x}^k + \overset{\smile}{\overset{\smile}{\Gamma}}{}^k_{\alpha\beta}\,\overset{\smile}{\dot{x}}{}^\alpha\,\dot{\overset{\smile}{x}}{}^\beta\ .$$

We have

$$\breve{x}^\alpha \circ \coprod = \breve{\breve{x}}^\alpha , \qquad \dot{x}^\alpha \circ \coprod = \ddot{x}^\alpha .$$

2.10. – Then we can introduce the covariant derivative in a way that, not making an essential use of free vectors, can be extended to manifolds.

Definition. Let $u \equiv (id_{\boldsymbol{E}}, \tilde{u}) \colon \boldsymbol{E} \to T\boldsymbol{E}$ and $v \equiv (id_{\boldsymbol{E}}, \tilde{v}) \colon \boldsymbol{E} \to T\boldsymbol{E}$ be C^∞ vector fields. The *covariant derivative* of v with respect to u is

$$\nabla_u v \equiv \coprod \circ \Gamma \circ Tv \circ u = \big(id_{\boldsymbol{E}}, D\tilde{u}(\tilde{v})\big) \colon \boldsymbol{E} \to T\boldsymbol{E} .$$

3. - Absolute kinematics

Here we introduce the basic elements of one-body kinematics independent of any frame of reference.

3.1. – *Absolute world-line and motion.* The basic definition of kinematics is the following. Here we consider a C^∞ world-line extending along the whole $\boldsymbol{T}$. We leave to the reader the easy generalization to the case when it is C^2 almost every where, or when it extends along an interval of $\boldsymbol{T}$.

Definition. A *world-line* is a connected C^∞ submanifold

$$\boldsymbol{M} \hookrightarrow \boldsymbol{E}$$

such that $\boldsymbol{S}_\tau \cap \boldsymbol{M}$ is a singleton, $\forall \tau \in \boldsymbol{T}$.

The *motion, relative to the world line* $\boldsymbol{M}$, is the map

$$M \colon \boldsymbol{T} \to \boldsymbol{E} , \qquad \text{given by} \qquad \tau \mapsto \text{the unique element} \in \boldsymbol{S}_\tau \cap \boldsymbol{M} .$$

Henceforth in this section we suppose a world-line $\boldsymbol{M}$, or its motion M, to be given.

3.2. – Proposition. $\boldsymbol{M}$ is an embedded 1-dimensional submanifold of $\boldsymbol{E}$, diffeomorphic to $\boldsymbol{R}$. M is a section of $(\boldsymbol{E}, t, \boldsymbol{T})$, namely it is a C^∞ embedding, such that

$$t \circ M = id_{\boldsymbol{T}} , \qquad \text{i.e. such that} \qquad M^0 \equiv x^0 \circ M = \underset{\frown}{x}^0 .$$

Hence the map

$$M\colon \boldsymbol{T} \to \boldsymbol{M} \qquad \text{is a } C^\infty \text{ diffeomorphism}\,.$$

The world line $\boldsymbol{M}$ is characterized by its motion M.

3.3. – The affine structures of $\boldsymbol{T}$ and $\boldsymbol{E}$ admit a kind of privileged world-lines.

Definition. $\boldsymbol{M}$ is *inertial* if it is an affine subspace of $\boldsymbol{E}$.

3.4. – Proposition. $\boldsymbol{M}$ is inertial if and only if M is an affine map, i.e.

$$M(\tau') = M(\tau) + DM(\tau' - \tau)\,, \qquad \text{with } DM \in \boldsymbol{U}\,.$$

3.5. – *Absolute velocity and acceleration.* Previously we introduce useful notations.

(a) Let $\boldsymbol{F}$ be an affine space and let $\varphi\colon \boldsymbol{T} \to \boldsymbol{F}$ be a C^∞ map. Then we put

$$\mathrm{d}\varphi \equiv (\varphi, D\varphi)\colon \boldsymbol{T} \to T\boldsymbol{F}\,.$$

In particular, if $\varphi \equiv f\colon \boldsymbol{T} \to \boldsymbol{E}$, we get

$$\mathrm{d}f \equiv (f, Df)\colon \boldsymbol{T} \to T\boldsymbol{E} \qquad \text{and} \qquad \mathrm{d}^2 f \equiv (f, Df, Df, D^2 f)\colon \boldsymbol{T} \to T^2\boldsymbol{E}\,.$$

(b) We put

$$\nabla \mathrm{d}f \equiv \coprod \circ \Gamma \circ \mathrm{d}^2 f = (f, D^2 f)\colon \boldsymbol{T} \to T\boldsymbol{E}\,.$$

The coordinate expressions are

$$\begin{aligned}
\mathrm{d}f &= Df^\alpha(\partial x_\alpha \circ f)\,,\\
\mathrm{d}^2 f &= Df^\alpha(\partial x_\alpha \circ \mathrm{d}f) + D^2 f^\alpha(\partial \dot{x}_\alpha \circ \mathrm{d}f)\,,\\
\nabla \mathrm{d}f &= \big(D^2 f^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ f)\, Df^\beta\, Df^\gamma\big)(\partial x_\alpha \circ f)\,.
\end{aligned}$$

3.6. – We can view the absolute velocity in terms of free or of applied vectors, equivalently.

Definition. The *free velocity* of M is the map $DM\colon \boldsymbol{T} \to \bar{\boldsymbol{E}}$.

The *velocity* of M is the map $\mathrm{d}M \equiv (M, DM)\colon \boldsymbol{T} \to T\boldsymbol{E}$.

3.7. – Proposition. We have

$$(*) \qquad \langle \underline{t}, DM \rangle = 1 .$$

Hence, we can write

$$DM\colon \boldsymbol{T} \to \boldsymbol{U} \qquad \text{and} \qquad \mathrm{d}M\colon \boldsymbol{T} \to \boldsymbol{V} \equiv \overset{|}{T}\boldsymbol{E}$$

and we get

$$DM^0 = 1 , \qquad DM = \delta x_0 \circ M + DM^k(\delta x_k \circ M) , \qquad \mathrm{d}M = \partial x_0 \circ M + DM^k(\partial x_k \circ M) .$$

Proof. $(*)$ follows from $t \circ M = id_{\boldsymbol{T}}$.

3.8. – We can view the absolute acceleration in terms of free or of applied vectors equivalently and second order tangent space may intervene esplicitly or not.

Definition. The *free acceleration* of M is the map $D^2M\colon \boldsymbol{T} \to \bar{\boldsymbol{E}}$.

The *lifted acceleration* of M is the map

$$\Gamma \circ \mathrm{d}^2 M = (M, DM; 0, D^2M)\colon \boldsymbol{T} \to \nu T^2 \boldsymbol{E} .$$

The *acceleration* of M is the map

$$\nabla \mathrm{d}M \equiv \coprod \circ \Gamma \circ \mathrm{d}^2 M = (M, D^2M)\colon \boldsymbol{T} \to T\boldsymbol{E} .$$

3.9. – Proposition. We have

$$(*) \qquad \langle \underline{t}, D^2M \rangle = 0 .$$

Hence, we can write

$$D^2M\colon \boldsymbol{T} \to \bar{\boldsymbol{S}} , \qquad \Gamma \circ \mathrm{d}^2M\colon \boldsymbol{T} \to \nu \overset{|}{T}{}^2 \boldsymbol{E} , \qquad \nabla \mathrm{d}M\colon \boldsymbol{T} \to \boldsymbol{A} \equiv \check{T}\boldsymbol{E}$$

and we get

$$\begin{aligned}
D^2M^0 &= 0 , \\
D^2M &= \big(D^2M^k + (\Gamma^k_{ij} \circ M)\, DM^i\, DM^j + 2(\Gamma^k_{0j} \circ M)\, DM^j + \Gamma^k_{00} \circ M\big)\, \delta x_k , \\
\Gamma \circ \mathrm{d}^2 M &= \big(D^2M^k + (\Gamma^k_{ij} \circ M)\, DM^i\, DM^j + 2(\Gamma^k_{0j} \circ M)\, DM^j + \Gamma^k_{00} \circ M\big)\, \partial \dot{x}_k , \\
\nabla \mathrm{d}M &= \big(D^2M^k + (\Gamma^k_{ij} \circ M)\, DM^i\, DM^j + 2(\Gamma^k_{0j} \circ M)\, Dm^j + \Gamma^k_{00} \circ M\big)\, \partial x_k .
\end{aligned}$$

3.10. – *Geometrical analysis.* Here we give some further element of analysis of M, not essential from a kinematical point of view.

M has two structures: the C^∞ structure induced by $\boldsymbol{E}$ and the oriented euclidean affine structure induced by $\boldsymbol{T}$ (but, in general, $\boldsymbol{M}$ is not an affine subspace of $\boldsymbol{E}$).

The embedding $TM\colon \boldsymbol{TT} \to \boldsymbol{TM} \to \boldsymbol{TE}$ is given by

$$(\tau, \lambda) \mapsto \big(M(\tau), \lambda\, DM(\tau)\big) .$$

The embedding $T^2M\colon T^2\boldsymbol{T} \to T^2\boldsymbol{M} \hookrightarrow T^2\boldsymbol{E}$ is given by

$$(\tau, \lambda; \mu, \nu) \to \big(M(\tau), \lambda\, DM(\tau); \mu\, DM(\tau), \nu\, DM(\tau) + \lambda\mu\, D^2M(\tau)\big) .$$

Now, let us consider the two fields

$$\bar{\boldsymbol{M}} \equiv \mathrm{d}M \circ M^{-1}\colon \boldsymbol{M} \to T\boldsymbol{M}, \qquad \text{and} \qquad \bar{\bar{\boldsymbol{M}}} \equiv \nabla \mathrm{d}M \circ M^{-1}\colon \boldsymbol{M} \to \check{T}\boldsymbol{E}_{|\boldsymbol{M}} .$$

3.11. – Proposition. $\bar{\boldsymbol{M}}$ results into the unitary oriented constant field, with respect to the oriented euclidean affine structure of $\boldsymbol{M}$ induced by π.

Moreover, each vector field $X\colon \boldsymbol{M} \to T\boldsymbol{M}$ can be written as $X = X^0 \bar{\boldsymbol{M}}$, where $X^0 \equiv \langle \underline{t}, X \rangle$.

3.12. – Proposition. Let $X\colon \boldsymbol{M} \to T\boldsymbol{M}$ and $Y\colon \boldsymbol{M} \to T\boldsymbol{M}$ be two C^∞ fields. Then the covariant derivative

$$\nabla_X Y \equiv \coprod \circ \Gamma \circ \ TY \circ X\colon \boldsymbol{M} \to T\boldsymbol{E}_{|\boldsymbol{M}}$$

is given by

$$\nabla_X Y = p^{\parallel}_{\bar{M}} \circ \nabla_X Y + p^{\perp}_{\bar{M}} \circ \nabla_X Y ,$$

where

$$p^{\parallel}_{\bar{M}} \circ \nabla_X Y = X^0\, D Y^0 \bar{\boldsymbol{M}} ,$$

results into the covariant derivative with respect to the affine structure of $\boldsymbol{M}$ and

$$p^{\perp}_{\bar{M}} \circ \nabla_X Y = X^0\, Y^0 \bar{\bar{\boldsymbol{M}}}$$

shows that the tensor

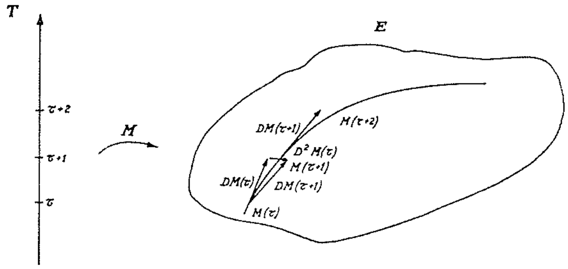
$$\bar{\bar{\boldsymbol{M}}} \otimes \underline{t} \otimes \underline{t}\colon \boldsymbol{M} \to T_{(1,2)}\boldsymbol{E}_{|\boldsymbol{M}}$$

can be considered as the second fundamental form of $\boldsymbol{M}$.

3.13. – *Physical description.* The world-line $\boldsymbol{M}$ of a particle represents the set of all the events « touched » by the particle and the motion M is the map that associates with each time the relative event. Of course the events being absolute, i.e. independent of any frame of reference, the same occurs for the world line and the motion. The affine structure of $\boldsymbol{E}$ allows a privileged type of motions, namely the inertial ones.

As we have the absolute motion M, we have the absolute velocity DM and acceleration D^2M. These contain all the information necessary to derive the velocity and acceleration observed by any frame of reference, when it is chosen. The fact that DM is a unitary vector and D^2M is a spatial vector will put in evidence how the observed velocity changes and that the observed acceleration does not change from an inertial frame of reference to an other.

We can describe the previous facts by pictures.



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Summary

The classical kinematical framework is constituted by a four dimensional affine space $\boldsymbol{E}$ *(absolute events), a three dimensional vector space* $\overline{\boldsymbol{S}} \subset \overline{\boldsymbol{E}}$ *(absolute simultaneity), an euclidean metric* $\check{g}$ *on* $\overline{\boldsymbol{S}}$ *(spatial metric) and an orientation on the one dimensional affine space* $\boldsymbol{T} \equiv \boldsymbol{E}/\boldsymbol{S}$ *(absolute oriented time). Then we get a bundle* $t\colon \boldsymbol{E} \to \boldsymbol{T}$ *which is trivial but not canonically a product.*

An absolute motion is a section $M\colon \boldsymbol{T} \to \boldsymbol{E}$. *Its absolute velocity is the first derivative* $\mathrm{d}M\colon \boldsymbol{T} \to T\boldsymbol{E}$ *and its absolute acceleration is the covariant derivative* $\nabla \mathrm{d}M\colon \boldsymbol{T} \to T\boldsymbol{E}$. *We get* $\langle \mathrm{d}t, \mathrm{d}M\rangle = 1$ *and* $\langle \mathrm{d}t, \nabla \mathrm{d}M\rangle = 0$.

In the following steps we shall analyse the absolute motion of a continuum, which determines a frame of reference, leading to the observed kinematics.

* * *