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# Approximation of evolution problems in Banach spaces (**)

A GIORGIO SESTINI per il suo 70° compleanno

## 0. - Introduction

Sequences of Banach spaces approximating a given Banach space $X$ were introduced by Trotter, [4], to show that a linear initial-value problem in $X$ can be interpreted, in some sense, as the limit of a suitable sequence of linear initial-value problems in spaces not necessarily coinciding with $X$.

In this paper, we prove that Trotter's method can be used to study semi-linear and non-linear initial-value problems. To this aim, we summarize some of Trotter's definitions and results in sect. **1** and some basic results on semi-linear and non-linear problems in sect. **2** and **3**. Trotter's method is then generalized to a semilinear problem with global (in time) solution in sect. **4** and to a semilinear problem with local (in time) solution in sect. **5**. Finally, a non-linear problem is studied in sect. **6**.

Most of our results can be proved with some formal (but not substantial) complications for more general semilinear and non-linear problems.

## 1. - Preliminaries: sequences of Banach spaces

Following Trotter, [4], we say that a sequence of Banach spaces ($B$-spaces) $\{X_n, n = 1, 2, 3, ...\}$ is a sequence of $B$-spaces approximating a given $B$-space $X$

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*if an operator $P_n$ exists such that*

$$P_n \in B(X, X_n) \quad \text{with} \quad \|P_n\| \leqslant 1\,, \ \forall n = 1, 2, \dots\,, \tag{1}$$

$$\lim_{n\to\infty} \|P_n f\|_n = \|f\| \qquad \forall f \in X\,, \tag{2}$$

*where $\|\cdot\|_n$ is a norm in $X_n$ and $\|\cdot\|$ is a norm in $X$.*

Now given a sequence $\{f_n\}$ with $f_n \in X_n$, $n = 1, 2, \dots$, by definition, we say that $\{f_n\}$ converges to $f \in X$ if

$$\lim_{n\to\infty} \|f_n - P_n f\|_n = 0\,. \tag{3}$$

Note that (3) implies $\lim_{n\to\infty} \|f_n\|_n = \|f\|$.

The following theorems can be proved on the approximation of a given semigroup in $X$ by means of a sequence of semigroups in $X_n$, [**4**], (notations are those of [**2**]$_1$).

Theorem 1. *Let $A$ a linear operator of class $G(M, \beta_0; X)$ and let $A_n$ be of class $G(M, \beta_0; X_n)$, $n = 1, 2, \dots$. If*

$$\lim_{n\to\infty} \|P_n R(z, A) f - R(z, A_n) P_n f\|_n = 0 \tag{4}$$

*for some $z$ with $\operatorname{Re} z > \beta_0$ and for each $f \in X$, where $R(z, A) = (zI - A)^{-1}$ and $R(z, A_n) = (zI - A_n)^{-1}$, then*

$$\lim_{n\to\infty} \|P_n \exp(tA) f - \exp(tA_n) P_n f\|_n = 0 \tag{5}$$

*uniformly in any finite interval of $t \geqslant 0$.*

Theorem 2. *If $A$, $A_n$ are as in Theorem 1 and if*

$$P_n[D(A)] \subset D(A_n) \qquad (n = 1, 2, \dots)\,, \tag{6a}$$

$$\lim_{n\to\infty} \|P_n A f - A_n P_n f\|_n = 0 \qquad \forall f \in D(A)\,, \tag{6b}$$

*then (4) is satisfied and (5) holds.*

Theorem 1 and 2 can be used to approximate the initial-value problems in $X$

$$(7)\qquad \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) \qquad (t>0)\,,\ A \in G(M, \beta_0; X)\,, \qquad X - \lim_{t\to 0^+} u(t) = u_0 \in D(A)\,,$$

by means of the sequence of the initial-value problems in $X_n$

$$(8)\qquad \frac{\mathrm{d}u_n(t)}{\mathrm{d}t} = A_n u_n(t) \qquad (t>0)\,,\ A_n \in G(M, \beta_0; X_n)\,, \qquad X_n - \lim_{t\to 0^+} u_n(t) = u_{n_0} \in D(A_n)\,,$$

where $\{X_n\}$ is a sequence of $B$-spaces approximating $X$.

In fact, if (5) holds and $u_{n_0} = P_n u_0$, then we have

$$(9)\qquad \lim_{n\to+\infty} \|P_n u(t) - u_n(t)\|_n = 0\,,$$

because $u(t) = \exp(tA)u_0$, $u_n(t) = \exp(tA_n)P_n u_0$.

Hence, $P_n u(t)$, the representation in $X_n$ of the solution of (7), is close to the solution $u_n(t)$ of the approximating problem (8) provided $n$ is large enaugh.

Since we intend to generalize the above theorems to semilinear and nonlinear initial-value problems, we list some existence and uniqueness results for such problems.

## 2. - The semilinear problem

If $A \in G(M, \beta_0; X)$ and $F = F(f)$ is a map from $X$ into $X$, the semilinear initial-value problem

$$(10a)\qquad \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)) \qquad (t>0)\,,$$

$$(10b)\qquad X\text{-}\lim_{t\to 0^+} u(t) = u_0\,,$$

is said to have the *strict* solution $u = u(t)$ over $[0, t_0]$ if

(a) $u(t) \in D(A) \cap D(F)$, $\forall t \in [0, t_0]$,

(b) $u(t)$ is strongly continuous $\forall t \in [0, t_0]$ and strongly differentiable $\forall t \in (0, t_0]$,

(c) $u(t)$ satisfies (10a) $\forall t \in [0, t_0]$ and (10b).

Theorem 3. *Under the assumptions*

— *$F$ has a domain $D(F)$ and range $R(F)$ contained in $X$ and an open and convex set $D \subset D(F)$ exists such that*

$$\|F(f) - F(f_1)\| \leqslant \alpha \|f - f_1\| \qquad \forall f, f_1 \in D\,, \tag{11a}$$

*where $\alpha$ is a non negative constant,*

— *$F(f)$ is Fréchet-differentiable at any $f \in D$ and its Fréchet-derivative $F_f$ is such that*

$$\|F_f g\| \leqslant \alpha_1 \|g\|\,, \qquad \forall f \in D\,,\ g \in X\,, \tag{11b}$$

*where $\alpha_1$ is a non negative constant that doesn't depend on $f$ and $g$,*

$$\|F_f g - F_{f_1} g\| \to 0 \quad \textit{as} \quad \|f - f_1\| \to 0 \quad \forall g \in X \textit{ with } f, f_1 \in D\,, \tag{11c}$$

$$u_0 \in D(A) \cap D\,, \tag{11d}$$

*then the semilinear initial-value problem* (10a) + (10b) *has a unique strict solution $u(t)$ over $[0, \hat{t}]$, provided that $\hat{t}$ is suitably small. Moreover, $u(t) \in D(A) \cap D$ $\forall t \in [0, \hat{t}]$.*

To prove Theorem 3, [**3**], one first shows that assumptions (11a), (11d) ensure the existence of a unique continuous solution $u(t) \in D$, $t \in [0, \hat{t}]$ (where $\hat{t}$ is suitably small) of the integral equation

$$u(t) = \exp(tA)\, u_0 + \int_0^t \exp\big((t-s)A\big)\, F\big(u(s)\big)\, ds\,, \tag{12}$$

that can be obtained by integrating formally system (10a) + (10b) over $[0, t]$.

Then, under the assumptions (11b), (11c), one proves that the continuous solution of (12) (which is called a *mild* solution of (10a) + (10b)) is the strict solution of the semilinear problem over $[0, \hat{t}]$.

This solution is obviously « local in time » because $u(t)$ is defined only over the small interval $[0, \hat{t}]$.

Now, if we want a unique strict solution $u(t)$ of (10a) + (10b) over a « large » interval $[0, t_0]$ given a priori, i.e. a *global* solution of (10a) + (10b), the following assumptions can be used

— *$F$ has a domain $D(F) = X$ and a range $R(F) \subset X$ and*

(13a) $$\|F(f) - F(f_1)\| \leqslant \alpha \|f - f_1\| \qquad \forall f, f_1 \in X,$$

*where $\alpha$ is a non negative constant or a non decreasing function of $\|f\|$ and $\|f_1\|$;*

— *$F(f)$ is Fréchet-differentiable at any $f \in D(F) = X$ and its Fréchet-derivative $F_f$ is such that*

(13b) $$\|F_f g\| \leqslant \alpha_1 \|g\| \qquad \forall f, g \in X,$$

*where $\alpha_1$ is a non negative constant or a non decreasing function of $\|f\|$;*

(13c) $$\|F_f g - F_{f_1} g\| \to 0 \qquad as \qquad \|f - f_1\| \to 0 \qquad \forall g, f, f_1 \in X,$$

(13d) $$u_0 \in D(A);$$

*if a strict solution $w = w(t)$ of* (10a) + (10b) *exists over* $[0, t_1] \subset [0, t_0]$, *then*

(13e) $$\|w(t)\| \leqslant \eta \qquad \forall t \in [0, t_1],$$

*where $\eta$ is a suitable constant depending* only *on $u_0$ and $t_0$ (and without loosing in generality such that $\|u_0\| \leqslant \eta$).*

Note that, if $\alpha$ is a constant, then (13e) follows from (13a).

### 3. - A non linear problem

Let us consider the following problem in the $B$-space $X$

(14) $$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) \qquad (t > 0),$$
$$X - \lim_{t \to 0^+} u(t) = u_0 \in D(A),$$

where $A$ is a non-linear operator with domain $D(A)$ and range $R(A)$ contained in $X$. If we assume that

(15) $\|f-g\| \leqslant \|f-g-z(Af-Ag)\| \qquad \forall f, g \in D(A), \ \forall z > 0,$

(16) $R(I-zA) = X, \qquad \forall z > 0$

(17) $X$ is an Hilbert space or a uniformly convex $B$-space,

then we have the following

Theorem 4. *Under the assumptions* (15)-(16)-(17), *for each* $u_0 \in D(A)$ *there exists an* $X$*-valued function* $u(t)$ *on* $[0, +\infty)$ *which satisfies* (14) *in the following sense*

(i) $u(t)$ *is strongly continuous for* $t \geqslant 0$ *and* $u(0) = u_0$,

(ii) $u(t) \in D(A)$ *for each* $t \geqslant 0$,

(iii) *the strong derivative* $\mathrm{d}u/\mathrm{d}t$ *exists and is strongly continuous except at a countable number of values* $t$ *and equals* $Au(t)$.

The function $u(t)$ solution of (14) in the sense of Theorem 4 is defined as follows

$$u(t) = X\text{-}\lim_{\nu \to \infty} u^{(\nu)}(t),$$

where $u^{(\nu)}(t)$ is the strict solution of the initial-value problem

$$\frac{\mathrm{d}u^{(\nu)}(t)}{\mathrm{d}t} = A^{(\nu)} u^{(\nu)}(t) \qquad (t > 0),$$

$$X - \lim_{t \to 0^+} u^{(\nu)}(t) = u_0,$$

with $A^{(\nu)} = \nu(J^{(\nu)} - \mathrm{I})$ and with $J^{(\nu)} = \big(1 - (\frac{1}{\nu} A)\big)^{-1}$ (see [**2**]$_2$).

## 4. - Approximation of a semilinear problem with global solution

Let $X$ a $B$-space and $\{X_n, n = 1, 2, \ldots\}$ a sequence of $B$-spaces approximating $X$.

Given the semilinear problem

$$\text{(18)}\qquad \begin{aligned} &\frac{du(t)}{dt} = Au(t) + F(u(t)) \qquad (t > 0)\,, \\ &X - \lim_{t \to 0^+} u(t) = u_0 \in D(A) \end{aligned}$$

with $A \in G(M, \beta_0; X)$ and $F = F(f)$ such that

$$\|F(f) - F(f_1)\| \leqslant \alpha \|f - f_1\| \qquad \forall f, f_1 \in X = D(F)\,,$$

where $\alpha$ is a non negative constant, consider the sequence of approximating problems

$$\text{(19)}\qquad \begin{aligned} &\frac{du_n(t)}{dt} = A_n u_n(t) + F_n(u_n(t)) \qquad (t > 0)\,, \\ &X_n - \lim_{t \to 0^+} u_n(t) = P_n u_0 \in D(A_n)\,, \end{aligned}$$

with $A_n \in G(M, \beta_0; X_n)$ and $F_n = F_n(g)$ such that

$$\|F_n(g) - F_n(g_1)\|_n \leqslant \beta \|g - g_1\|_n \qquad \forall g, g_1 \in X_n = D(F_n)\,,$$

where $\beta$ is a non negative constant that doesn't depend on n. We have the following

Theorem 5. *Under the above assumptions on $A$, $A_n$, $F$, $F_n$, if*

$$\text{(20)}\qquad \lim_{n \to \infty} \|Z_n(t) P_n f - P_n Z(t) f\|_n = 0 \qquad \forall f \in X$$

*uniformly in any finite interval of $t \geqslant 0$, with $Z(t) = \exp(tA)$ and $Z_n(t) = \exp(tA_n)$ and if*

$$\text{(21)}\qquad \lim_{n \to \infty} \|F_n(P_n f) - P_n F(f)\|_n = 0 \qquad \forall f \in X\,,$$

*then the mild solutions $u(t)$ and $u_n(t)$ respectively of* (18) *and* (19) *are such that*

$$\text{(22)}\qquad \lim_{n \to \infty} \|P_n u(t) - u_n(t)\|_n = 0\,,$$

$\forall t \in [0, t_0]$, *where* $t_0$ *is arbitrarly chosen with* $0 < t_0 < +\infty$.

Proof. We have for any $t \in [0, t_0]$

$$\|P_n u(t) - u_n(t)\|_n \leqslant \|P_n Z(t) u_0 - Z_n(t) P_n u_0\|_n \tag{23}$$

$$+ \int_0^t \{\|P_n Z(t-s) F(u(s)) - Z_n(t-s) P_n F(u(s))\|_n + M \exp(\beta_0(t-s)) \cdot$$

$$\cdot (\|P_n F(u(s)) - F_n(P_n u(s))\|_n + \|F_n(P_n u(s)) - F_n(u_n(s))\|_n)\} \, ds \, .$$

If we put

$$w_n(t) = \|P_n u(t) - u_n(t)\|_n \, ,$$

$$\varphi_n(t) = \|P_n Z(t) u_0 - Z_n(t) P_n u_0\|_n + \int_0^t \|P_n Z(t-s) F(u(s))$$

$$- Z_n(t-s) P_n F(u(s))\|_n + \mathrm{M} \exp(\beta_0(t-s)) \|P_n F(u(s)) - F_n(P_n u(s))\|_n \, ,$$

then we obtain from (23)

$$w_n(t) \leqslant \varphi_n(t) + M\beta \int_0^t \exp(\beta_0(t-s)) w_n(s) \, ds \, . \tag{24}$$

To prove that $\lim_{n\to\infty} \varphi_n(t) = 0 \quad \forall t \in [0, t_0]$, we first note that (20) implies

$$\lim_{n\to\infty} \|P_n Z(t) u_0 - Z_n(t) P_n u_0\|_n = 0$$

uniformly with respect to $t \in [0, t_0]$ and

$$\lim_{n\to\infty} \|P_n Z(t-s) F(u(s)) - Z_n(t-s) P_n F(u(s))\|_n = 0 \, .$$

Then, by using the theorem of dominated convergence we have

$$\lim_{n\to\infty} \int_0^t \{\|P_n Z(t-s) F(u(s)) - Z_n(t-s) P_n F(u(s))\|_n$$

$$+ M \exp(\beta_0(t-s)) \|P_n F(u(s)) - F_n(P_n u(s))\|_n\} \, ds = 0 \, ,$$

because for each $n \geqslant \bar{n}$

$$\|P_n Z(t-s) F(u(s)) - Z_n(t-s) P_n F(u(s))\|_n +$$

$$+ \mathrm{M} \exp\big(\beta_0(t-s)\big) \|P_n F(u(s)) - F_n(P_n u(s))\|_n$$

$$\leqslant 3M(\alpha k + \|F(\theta_X)\|) \exp\big(\beta_0(t-s)\big) + M \exp\big(\beta_0(t-s)\big)\big(\beta k + 1 + \|F(\theta_X)\|\big) ,$$

where $\bar{n}$ is such that $\|F_n(P_n \theta_X) - P_n F(\theta_X)\|_n \leqslant 1$, and $k$ is a constant such that $\|u(s)\| \leqslant k \quad \forall s \in [0, t] \subset [0, t_0]$. Hence, $\lim_{n\to\infty} \varphi_n(t) = 0 \quad \forall t \in [0, t_0]$.

On the other hand if

$$\eta_n(t) = w_n(t) \exp(-\beta_0 t) , \qquad g_n(t) = \varphi_n(t) \exp(-\beta_0 t) ,$$

(24) becomes

$$0 \leqslant \eta_n(t) \leqslant g_n(t) + M\beta \int_0^t \eta_n(s)\, \mathrm{d}s .$$

Using Gromwall's inequality, [**1**], we obtain

$$\eta_n(t) \leqslant g_n(t) + M\beta \int_0^t g_n(s) \exp\big(M\beta(t-s)\big)\, \mathrm{d}s .$$

Since $\lim_{n\to\infty} \varphi_n(t) = 0$, we have finally

$$\lim_{n\to\infty} \eta_n(t) = 0 , \qquad \text{i.e.} \quad \lim_{n\to\infty} \|P_n u(t) - u_n(t)\|_n = 0 \qquad \forall t \in [0, t_0] .$$

Corollary 1. *Under the assumptions of Theorem* 5, *and if assumptions* (13b), (13c) *are satisfied by* $F(f)$ *and by* $F_n(g)$ *with* $X_n$ *instead of* $X$, *then* (22) *holds, where* $u(t)$ *and* $u_n(t)$ *are the* g l o b a l *solution of* (18) *and* (19).

Proof. Our assumptions ensure that the mild solutions of (18) and (19) are the global solutions of (18) and (19) over $[0, t_0]$.

Remark. (20) is certainly satisfied if (4) or (6a) + (6b) hold.

## 5. - Approximation of a semilinear problem with local solution.

Now, consider the semilinear problem in the $B$-space $X$

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)) \qquad (t > 0)\,, \tag{25a}$$

$$\lim_{t \to 0^+} u(t) = u_0 \in D(A) \cap D\,, \tag{25b}$$

where $A \in G(M, \beta_0; X)$ and $F = F(f)$ is such that

$$D(F) \subset X\,, \qquad R(F) \subset X\,, \tag{26a}$$

an open and convex set $D \subset D(f)$ exists such that

$$\|F(f) - F(f_1)\| \leqslant \alpha \|f - f_1\| \qquad \forall f, f_1 \in D \tag{26b}$$

where $\alpha$ is a non negative constant.

Correspondingly, the approximating problem reads as follows

$$\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} = A_n u_n(t) + F_n(u_n(t)) \qquad (t > 0)\,, \tag{27a}$$

$$\lim_{t \to 0^+} u_n(t) = P_n u_0 \in D(A_n) \cap D_n\,, \tag{27b}$$

where $A_n \in G(M, \beta_0; X_n)$ and $F_n = F_n(g)$ is such that

$$D(F_n) \subset X_n\,, \qquad R(F_n) \subset X_n\,, \tag{28a}$$

an open and convex set $D_n \subset D(F_n)$ exists such that

$$\|F_n(g) - F_n(g_1)\|_n \leqslant \beta \|g - g_1\|_n \qquad \forall g, g_1 \in D_n\,, \tag{28b}$$

where $\beta$ is a non negative constant which doesn't depend on $n$.

Theorem 6. *If conditions* (26a, b), (28a, b) *are satisfied and*

$$\lim_{n \to \infty} \|P_n Z(t) f - Z_n(t) P_n f\|_n = 0 \qquad \forall f \in X\,, \tag{29}$$

*uniformly in any finite interval of* $t \geqslant 0$, *and if*

$$P_n[D] \subset D_n\,, \tag{30}$$

$$\lim_{n\to\infty} \|P_n F(f) - F_n(P_n f)\|_n = 0 \qquad \forall f \in D\,, \tag{31}$$

*then the mild solution* $u(t)$ *and* $u_n(t)$ *of* (25a) + (25b) *and* (27a) + (27b) *respectively are such that*

$$\lim_{n\to\infty} \|P_n u(t) - u_n(t)\|_n = 0 \qquad \forall t \in [0, \tilde{t}]\,, \tag{32}$$

*where* $\tilde{t}$ *is suitably small and doesn't depend on* $n$.

Proof. Problem (25) has a unique mild solution $u(t) \in D$, $\forall t \in [0, \hat{t}]$, where $\hat{t}$ is chosen suitably small and such that

$$q(\hat{t}) < 1\,, \tag{33}$$

with

$$q(\hat{t}) = \frac{1}{r}\max\left(\|Z(t)u_0 - u_0\|,\, t \in [0, \hat{t}]\right) + \frac{M}{r}\,\frac{\exp(\beta_0 \hat{t}) - 1}{\beta_0}\,(\gamma r + \|F(u_0)\|)\,, \tag{34}$$

where $\gamma = \max(\alpha, \beta)$ and $r$ is the radius of a sphere in $D$.

In an analogous way, (27a) + (27b) has a unique mild solution $u_n(t) \in D_n$, $\forall t \in [0, \hat{t}_n]$, where $\hat{t}_n$ is chosen such that

$$q_n(\hat{t}_n) < 1 \tag{35}$$

where

$$q_n(\hat{t}_n) = \frac{1}{r}\max\left\{\|Z_n(t)P_n u_0 - P_n u_0\|_n,\, t \in [0, \hat{t}_n]\right\} + \frac{M}{r}\,\frac{\exp(\beta_0 \hat{t}_n) - 1}{\beta_0}\,(\gamma r + \|F_n(P_n u_0)\|_n)\,. \tag{36}$$

Now, choose $\varepsilon \in (0, 1)$ and take $\tilde{t} > 0$ such that $p(\tilde{t}) < 1$, where

$$p(\tilde{t}) = \frac{1}{r}\{\varepsilon + \max\left(\|Z(t)u_0 - u_0\|\ t \in [0, \tilde{t}]\right)\} + \frac{M}{r}\frac{\exp(\beta_0 \tilde{t}) - 1}{\beta_0}(\gamma r + \varepsilon + \|F(u_0)\|)\,.$$

Then, we have from (33) and (34) that $\tilde{t} < \hat{t}$, whereas (36) with $\tilde{t}$ instead of $\hat{t}_n$ gives

$$q_n(\tilde{t}) = \frac{1}{r}\max\{\|Z_n(t)P_n u_0 - u_0\|_n\ t \in [0, \tilde{t}]\} + \frac{M}{r}\frac{\exp(\beta_0 \tilde{t}) - 1}{\beta_0}(\gamma r + \|F_n(P_n u_0)\|_n)$$

$$\leqslant \frac{1}{r}\max\{\|Z_n(t)P_n u_0 - P_n Z(t)u_0\|_n + \|P_n Z(t)u_0 - P_n u_0\|_n\ t \in [0, \tilde{t}]\} + \frac{M}{r}\frac{\exp(\beta_0 \tilde{t}) - 1}{\beta_0}(\gamma r + \|F_n(P_n u_0) - P_n F(u_0)\|_n + \|P_n F(u_0)\|_n).$$

By using (29), (31) it follows that an integer $\bar{n} = \bar{n}(\varepsilon)$ exists such that for $n \geqslant \bar{n}$

$$\|Z_n(t)P_n u_0 - P_n Z(t)u_0\|_n < \varepsilon\,, \qquad \|F_n(P_n u_0) - P_n F(u_0)\|_n < \varepsilon\,.$$

Therefore, by using (1), we have for $n \geqslant \bar{n}$

$$q_n(\tilde{t}) \leqslant \frac{1}{r}\{\varepsilon + \max\left[\|Z(t)u_0 - u_0\|\ t \in [0, \tilde{t}]\right]\} + \frac{M}{r}\frac{\exp(\beta_0 \tilde{t}) - 1}{\beta_0}(\gamma r + \varepsilon + \|F(u_0)\|) = p(\tilde{t}) < 1\,.$$

Hence, for a fixed $\varepsilon > 0$ (and, therefore, for a suitably small $\tilde{t} > 0$), we obtain that (25a) + (25b) and (27a) + (27b) with $n \geqslant \bar{n}$ have mild solution $u(t)$ and $u_n(t)$ respectively where $t \in [0, \tilde{t}]$ and $\tilde{t}$ *doesn't* depend on $n$.

To prove that $\lim_{n\to\infty} \|P_n u(t) - u_n(t)\|_n = 0$; it is enough to take into account that $u(t) \in D$ and $u_n(t) \in D_n$ for $n \geqslant \bar{n}$, $\forall t \in [0, \tilde{t}]$ and then we can proceed as in the proof of Theorem 5.

Corollary 2. *Under the assumptions of Theorem* 6, *and if* (11b) *and* (11c) *hold for $F$ and $F_n$, then*

$$\lim_{n\to+\infty} \|P_n u(t) - u_n(t)\|_n = 0 \qquad \forall t \in [0, \bar{t}] , \tag{37}$$

*where $u(t)$ and $u_n(t)$ are the strict solutions of* (25a) + (25b) *and* (27a) + (27b) *and $\bar{t}$ is suitably small and doesn't depend on $n$.*

Proof. See the proof of Corollary 1.

Remark. If we have instead of (25a)

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t), t) \tag{25a'}$$

and so, instead of (27a)

$$\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} = A_n u_n(t) + F_n(u_n(t), t) , \tag{27a'}$$

Theorem 6 and Corollary 2 are still true if suitable assumptions involving the Fréchet-derivatives of $F$ and $F_n$ are satisfied.

## 6. - Approximation of a non-linear problem

Consider the following evolution problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) , \qquad X - \lim_{t\to 0^+} u(t) = u_0 \in D(A) , \tag{40}$$

where $A$ is a nonlinear operator with domain $D(A) \subset X$ and range $R(A) \subset X$.

If assumptions (15) and (16) are satisfied and the $B$-space $X$ is uniformly convex (or it is an Hilbert space), Theorem 4 holds and so $u(t) = X\text{-}\lim_{\nu\to\infty} u^{(\nu)}(t)$ uniformly with respect to $t \in [0, \bar{t}]$, $0 < \bar{t} < +\infty$.

Let $\{X_n\}$ be a sequence of $B$-spaces approximating $X$, and consider the approximating problem

$$\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} = A_n u_n(t) \ \ (t > 0) , \qquad X_n - \lim_{t\to 0^+} u_n(t) = P_n u_0 \in D(A_n) , \tag{41}$$

where $A_n$ are nonlinear operators with domain $D(A_n) \subset X_n$ and range $R(A_n) \subset X_n$ ($X_n$ are also uniformly convex spaces).

Moreover, assume that

$$\|f - g\|_n \leqslant \|f - g - z(A_n f - A_n g)\|_n \qquad \forall f, g \in D(A_n), \ z > 0, \tag{42}$$

$$R(I - zA_n) = X_n \qquad \forall z > 0. \tag{43}$$

Then, Theorem 4 holds for each problem (41) and so an $X_n$-valued function $u_n(t)$ exists, which is a strict solution of (41) except at a countable number of values of $t$.

We also have that $u_n(t) = X_n\text{-}\lim_{\nu\to\infty} u_{n}^{(\nu)}(t)$ uniformly in any finite interval of $t$, where $u_n^{(\nu)}(t)$ are the strict solution of

$$\frac{du_n^{(\nu)}(t)}{dt} = A_n^{(\nu)} u_n^{(\nu)}(t) \quad (t > 0), \quad X_n - \lim_{t\to 0^+} u_n^{(\nu)}(t) = P_n u_0 \tag{44}$$

$$A_n^{(\nu)} = \nu(J_n^{(\nu)} - I), \quad J_n^{(\nu)} = (I - \frac{1}{\nu} A_n)^{-1}, \quad D(A_n^{(\nu)}) = X_n, \tag{45}$$

$$\|A_n^{(\nu)} f - A_n^{(\nu)} g\|_n \leqslant 2\nu \|f - g\|_n. \tag{46}$$

Theorem 7. *If* (15), (16), (42), (43) *hold and moreover*

$$P_n[D(A)] \subset D(A_n) \qquad (n = 1, 2, 3, \ldots), \tag{47a}$$

$$\lim_{n\to\infty} \|P_n A f - A_n P_n f\|_n = 0 \qquad \forall f \in D(A), \tag{47b}$$

*then the strict solution* $u = u(t)$ *and* $u_n = u_n(t)$ *of* (40) *and* (41) (*in the sense of Theorem* 4) *are such that*

$$\lim_{n\to\infty} \|P_n u(t) - u_n(t)\|_n = 0 \tag{48}$$

*for any* $t \in [0, \bar{t}]$, *where* $0 < \bar{t} < +\infty$.

To prove this theorem, we state the following

Lemma 1. *If the assumptions of Theorem* 7 *are satisfied we have*

$$\lim_{n\to\infty} \|A_n^{(\nu)} P_n g - P_n A^{(\nu)} g\|_n = 0 \qquad \forall g \in X, \ \forall \nu. \tag{49}$$

Proof. Taking into account (45) we obtain

$$\|J_n^{(\nu)}P_n g - P_n J^{(\nu)} g\|_n \leqslant \frac{1}{\nu}\|A_n P_n f - P_n A f\|_n ,$$

with $f = (I-(1/\nu)A)^{-1}g \in D(A)$ for any $g \in X$.

Hence, $\forall g \in X$

$$\|A_n^{(\nu)}P_n g - P_n A^{(\nu)} g\|_n = \nu\|J_n^{(\nu)}P_n g - P_n J^{(\nu)} g\|_n \leqslant \|A_n P_n f - P_n A f\|_n ,$$

where $f \in D(A)$ and (49) follows from (47*b*).

Now, we can prove Theorem 7. Given $\varepsilon > 0$, we have that a $\nu_\varepsilon$ exists such that for $\nu \geqslant \nu_\varepsilon$

$$\|u^{(\nu)}(t) - u(t)\| < \varepsilon \tag{50}$$

(because the sequence $\{u^{(\nu)}(t)\}$ converges to $u(t)$ uniformly $\forall t \in [0, \bar{t}]$).

Moreover, the following condition holds for the sequence $\{u_n^{(\nu)}(t)\}$ (see [**2**]$_2$)

$$\|u_n^{(\nu)}(t) - u_n(t)\|_n^2 \leqslant 4\|A_n P_n u_0\|_n^2 \frac{\bar{t}}{\nu} \qquad \forall t \in [0, \bar{t}] . \tag{51}$$

However,

$$\begin{aligned}\|u_n^{(\nu)}(t) - u_n(t)\|_n^2 &\leqslant 4(\|A_n P_n u_0 - P_n A u_0\|_n + \|P_n A u_0\|_n)^2 \frac{\bar{t}}{\nu} \\ &\leqslant 4(\|A_n P_n u_0 - P_n A u_0\|_n + \|A u_0\|)^2 \frac{\bar{t}}{\nu} .\end{aligned} \tag{52}$$

We have from (47b) that an integer $\bar{n}$ exists such that for $n \geqslant \bar{n}$ $\|A_n P_n \cdot u_0 - P_n A u_0\|_n < 1$, and so, (52) becomes for any $n \geqslant \bar{n}$ ($\bar{n}$ may depends on $u_0$)

$$\|u_n^{(\nu)}(t) - u_n(t)\|_n^2 \leqslant 4(1 + \|A u_0\|)^2 \frac{\bar{t}}{\nu} \quad \forall t \in [0, \bar{t}] , \quad \nu = 1, 2, 3, \ldots .$$

Then, $\nu'_\varepsilon$ exists such that for $\nu \geqslant \nu'_\varepsilon$

$$\|u_n(t) - u_n^{(\nu)}(t)\|_n < \varepsilon \qquad \forall t \in [0, \bar{t}] , \tag{53}$$

for each $n \geqslant \bar{n}$. Note that $\nu'_\varepsilon$ depends on $\varepsilon$ and does not on $n$. Hence, if we put $\mu = \max(\nu_\varepsilon, \nu'_\varepsilon)$, both (50) and (53) hold (for $\nu = \mu$ and $n \geqslant \bar{n}$).

Then

$$\|P_n u(t) - u_n(t)\|_n \leqslant \|P_n u(t) - P_n u^{(\mu)}(t)\|_n$$

$$+ \|P_n u^{(\mu)}(t) - u^{(\mu)}(t)\|_n + \|u_n^{(\mu)}(t) - u_n(t)\|_n$$

$$\leqslant 2\varepsilon + \|P_n u^{(\mu)}(t) - u_n^{(\mu)}(t)\|_n .$$

However, we obtain from Theorem 5 with $A = A_n = 0$, $F = A^{(\mu)}$ and $F_n = A_n^{(\mu)}$

$$\|P_n u^{(\mu)}(t) - u_n^{(\mu)}(t)\|_n < \varepsilon \quad \text{for } n \geqslant n_\varepsilon \qquad \forall t \in [0, \bar{t}] .$$

Therefore, given $\varepsilon > 0$, we obtain for $n > \max(\bar{n}, n_\varepsilon)$

$$\|P_n u(t) - u_n(t)\|_n < 3\varepsilon \qquad \forall t \in [0, \bar{t}] ,$$

i.e. (48) is proved.

## 7. - Examples

Example 1. Consider the $B$-space

$$X = \{f\colon f(x) \in C[a, b], f(a) = 0\} \tag{54}$$

with norm $\|f\| = \max\{|f(x)| \, x \in [a, b]\}$.

If $X_n$ is the $B$-space of all ordered real $n$-tuples, defined as follows

$$X_n = \{f_n\colon f_n = (0, f_n^2, \ldots, f_n^n)\} \tag{55}$$

with norm $\|f_n\|_n = \max\{|f_n^i| \, i = 1, 2, \ldots, n\}$ it is easy to show that the sequence $\{X_n\}$ is a sequence of $B$-spaces approximating $X$ with

$$P_n f = \big(0, f(x_2), \ldots, f(x_n)\big) \qquad \forall f \in X ,$$

where $x_i = a + (i-1)\delta_n$ $(i = 1, 2, \ldots, n)$, $\delta_n = (b-a)/(n-1)$ $(n = 2, \ldots)$.

If we define the following operators

$$Af = -v\frac{\mathrm{d}f}{\mathrm{d}x}\, D(A) = \{f\colon f(x), \frac{\mathrm{d}f}{\mathrm{d}x} \in X\} , \tag{56}$$

where $v$ is a positive constant

$$(57) \quad A_n f_n = -\frac{v}{\delta_n}(0, f_n^2, \ldots, f_n^i - f_n^{i-1}, \ldots, f_n^n - f_n^{n-1}) \qquad \forall f_n \in D(A_n) = X_n$$

$$(58) \quad F(f) = f^2 \qquad \forall f \in X\,,\ D(F) = X\,,\ R(F) \subset X\,,$$

$$(59) \quad F_n(f_n) = F_n(0, f_n^2, \ldots, f_n^n) = \big(0, (f_n^2)^2, \ldots, (f_n^n)^2\big)\,,$$

$$D(F_n) = X_n\,, \qquad R(F_n) \subset X_n\,,$$

it is possible to show that the following holds

Theorem 8. *By using* (56), (57), (58), (59) *we have*

(a) $A \in G(1, 0; X)$, $A_n \in G(1, 0; X_n)$;

(b) (11b)-(11c)-(11d) *hold for* $F$ *and* $F_n$.

Thus, the evolution problem in $X$

$$(60) \qquad \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + u^2(t) \quad (t > 0)\,, \qquad X - \lim_{t \to 0^+} u(t) = u_0 \in D(A)\,,$$

where $A$ is defined by (56) can be approximated (in the sense of Theorem 6 and Corollary 2) by means of the problems in $X_n$

$$(61) \qquad \frac{\mathrm{d}u_n(t)}{\mathrm{d}t} = A_n u_n(t) + u_n^2(t) \quad (t > 0)\,, \qquad X_n - \lim_{t \to 0^+} u_n(t) = P_n u_0 \in D(A_n)\,,$$

where $A_n$ is defined by (57).

Example 2. We can apply again Theorem 6 and Corollary 2 if $X$, $X_n$, $A$, $A_n$ are as in Example 1 and

$$F(f)(x) = \int_a^x f^2(y)\,\mathrm{d}y\,, \qquad D(F) \subset X\,,\ R(F) \subset X\,;$$

$$F_n(f_n) = \delta_n(0, \ldots, g_n^i, \ldots, g_n^n) \qquad \forall f_n \in D(F_n) = X_n\,,$$

where $g_n^i = \sum_{j=1}^{i} (f_n^j)^2$.

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## Summary

*We show that Trotter's method of approximating sequences of Banach spaces can be used to study semilinear and non-linear initial-value problems.*

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