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Particle transport in a slab bounded by capturing and reflecting planes (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

Consider the following integro-differential problem

$$(1)\qquad \frac{\partial}{\partial t}u(x,y;t) = -vy\frac{\partial}{\partial x}u(x,y;t) - v\sigma u(x,y;t) + $$
$$+\tfrac{1}{2}v\sigma_s\int_{-1}^{+1}u(x,y';t)\,dy' + q(x,y;t)\,, \qquad |x|<a,\ |y|<1,\ t>0,$$

$$(2a)\qquad yu(-a,y;t) = \int_{-1}^{0}h(y,y')|y'|u(-a,y';t)\,dy', \qquad y\in(0,1),\ t>0,$$

$$(2b)\qquad |y|u(a,y;t) = \int_{0}^{1}k(y,y')y'u(a,y';t)\,dy', \qquad y\in(-1,0),\ t>0,$$

$$(3)\qquad u(x,y;0) = u_0(x,y)\,, \qquad |x|<a,\ |y|<1,$$

where a and v are positive constants; σ and σ_s are nonnegative constants; $q(x,y;t)$ and $u_0(x,y)$ are given nonnegative summable functions; $h(y,y')$ and

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$k(y, y')$ are nonnegative functions, such that

$$(4) \qquad \begin{aligned} &\int_0^1 h(y, y')\,\mathrm{d}y = \bar{h}(y') \leqslant b \leqslant 1 && \forall y' \in (-1, 0)\,, \\ &\int_{-1}^0 k(y, y')\,\mathrm{d}y = \bar{k}(y') \leqslant b \leqslant 1 && \forall y' \in (0, 1)\,, \end{aligned}$$

[**2**], [**3**], [**5**]. System (1)-(3) is a one-group particle-transport problem in a homogeneous slab of thickness 2a ($< \infty$), under the assumptions of plane symmetry and isotropic scattering. The unknown function $u(x, y; t)$ is a particles density (e.g., a photon density), i.e., $u(x, y; t)\,\mathrm{d}x\,\mathrm{d}y$ is the number of particle that, at time t, are between x and $x + \mathrm{d}x$ and are such that the cosine of the angle between their velocity $\boldsymbol{v}$ and the positive x axis is between y and $y + \mathrm{d}y$. Moreover, $q(x, y; t)$ is a source term, $u_0(x, y)$ is the initial particle density, and σ, σ_s are cross sections that charaterize the physical properties of the materials of the slab. Finally, the boundary conditions (2) show that particles are either captured or reflected by the boundary planes $x = -a$ and $x = a$, (for instance, $h(y, y')\,\mathrm{d}y\,\mathrm{d}y'$ is the probability that a particle at $x = -a$ and such that the cosine between its velocity $\boldsymbol{v}$ and the positive x-axis is between y' and $y' + \mathrm{d}y'$ with $y' \in (-1, 0)$ is reflected by the plane $x = -a$ and emerges with a cosine between y and $y + \mathrm{d}y$, with $y \in (0, 1)$).

Remark 1. If

$$(4)' \qquad \begin{aligned} &\bar{h}(y') = \int_0^1 h(y, y')\,\mathrm{d}y = 1 && \forall y' \in (-1, 0)\,, \\ &\bar{k}(y') = \int_{-1}^0 k(y, y')\,\mathrm{d}y = 1 && \forall y' \in (0, 1)\,, \end{aligned}$$

then the boundary planes do not capture particles because

$$\int_0^1 y u(-a, y; t)\,\mathrm{d}y = \int_{-1}^0 |y'|\,u(-a, y'; t)\,\mathrm{d}y'\,,$$

$$\int_{-1}^0 |y|\,u(a, y; t)\,\mathrm{d}y = \int_0^1 y' u(a, y'; t)\,\mathrm{d}y'\,.$$

Since

$$(5) \qquad N(t) = \int_{-1}^{+1} \mathrm{d}y \int_{-a}^{+a} u(x, y; t)\,\mathrm{d}x$$

is the total number of particles in the slab at time t, and $u(x, y; t)$ is a particle density (i.e., a nonnegative function), we introduce the (real) Banach space

$$X = L^1((-a, a)\times(-1, 1)), \quad \|f\| = \int_{-1}^{+1} \mathrm{d}y \int_{-a}^{+a} |f(x, y)|\,\mathrm{d}x$$

and the (closed) positive cone of X

$$X_0 = \{f\colon f\in X;\ f(x, y)\geqslant 0 \text{ for a.e. } (x, y)\in(-a, a)\times(-1, 1)\}.$$

To write system (1)-(3) as a problem of evolution in the space X, we define the operators

$$A_0 f = -vy\,\partial f/\partial x, \quad D(A_0) = \{f\colon f\in X;\ y\,\partial f/\partial x\in X;\ f \text{ satisfies the boundary conditions (2)}\}, \tag{6}$$

$$A = A_0 - v\sigma I, \quad D(A) = D(A_0), \tag{7}$$

$$Jf = \tfrac{1}{2}\int_{-1}^{+1} f(x, y')\,\mathrm{d}y', \quad D(J) = X, \tag{8}$$

where $\partial f/\partial x$ is a distributional derivative. Then, (1)-(3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + v\sigma_s Ju(t) + q(t) \quad (t>0), \qquad \lim_{t\to 0_+}\|u(t) - u_0\| = 0, \tag{9}$$

where $u(t) = u(.\,, .\,; t)$ and $q(t) = q(.\,, .\,; t)$ are now to be interpreted as functions from $[0, +\infty)$ into X (or into X_0), $\mathrm{d}u/\mathrm{d}t$ is a strong derivative, and it is assumed that u_0 is a given element of $D(A)\cap X_0$. Moreover, $u(t)$, $t\in[0, +\infty)$, is said to be a (strict) solution of (9) if (i) $u(t)$ is continuously differentiable at any $t\geqslant 0$, (ii) $u(t)\in D(A)$ $\forall t\geqslant 0$, (iii) $u(t)$ satisfies (9).

2. - The operators J and A_0

The following lemmas are needed to prove that system (9) has a unique strict solution.

Lemma 1. (a) $J\in\mathscr{B}(X)$, [1], [4], *with* $\|Jf\|\leqslant\|f\|$ $\forall f\in X$; (b) $Jf\in X_0$ *and* $\|Jf\| = \|f\|$ $\forall f\in X_0$.

Proof. (a) We have from (8)

$$\|Jf\| = \int_{-1}^{+1} dy \int_{-a}^{+a} |\tfrac{1}{2} \int_{-1}^{+1} f(x, y')\, dy'|\, dx \leqslant \int_{-a}^{+a} dx \int_{-1}^{+1} |f(x, y')|\, dy' = \|f\| \quad \forall f \in X .$$

(b) follows directly from (8).

Lemma 2. (a) *The operator* $(zI - A_0)^{-1}$ *exists and belongs to* $\mathscr{B}(X)$ *for any* $z > 0$; (b) $(zI - A_0)^{-1} g \in X_0$, $\forall g \in X_0$, $z > 0$.

Proof. (a) If $g \in X$ and $z > 0$ are given, consider the equation

$$(zI - A_0) f = g , \tag{10}$$

where the unknown f must obviously be sought in $D(A_0)$. Since (10) can be written as follows

$$\frac{\partial}{\partial x} f(x, y) + \frac{z}{vy} f(x, y) = \frac{1}{vy} g(x, y) \qquad \text{for a.e. } (x, y) \in (-a, a) \times (-1, 1) ,$$

we have

$$\text{(11a)} \qquad f(x, y) = \frac{-1}{vy} \Big[C_1(y) \exp \Big[\frac{z}{vy} (a - x)\Big] + \int_x^a \exp \Big[\frac{-z}{vy} (x - x')\Big]\, g(x', y)\, dx'\Big], \qquad y \in (-1, 0) ,$$

$$\text{(11b)} \qquad f(x, y) = \frac{1}{vy} \Big[C_2(y) \exp \Big[\frac{-z}{vy} (a + x)\Big] + \int_{-a}^x \exp \Big[\frac{-z}{vy} (x - x')\Big]\, g(x', y)\, dx'\Big], \qquad y \in (0, 1) ,$$

where $C_1(y)$ and $C_2(y)$ are to be chosen so that $f(x, y)$ satisfies the boundary conditions (2). Thus,

$$\text{(12a)} \qquad C_1(y) = \int_0^1 k(y, y') \exp(-2az/vy')\, C_2(y')\, dy' + \int_0^1 k(y, y') G_2(y')\, dy' , \qquad y \in (-1, 0) ,$$

$$(12b)\qquad C_2(y) = \int_{-1}^{0} h(y, y') \exp(2az/vy')\, C_1(y')\, dy' + \int_{-1}^{0} h(y, y')\, G_1(y')\, dy', \qquad y \in (0, 1),$$

where

$$(13)\qquad \begin{aligned} G_1(y) &= \int_{-a}^{+a} \exp\left[\frac{z}{vy}(a + x')\right] g(x', y)\, dx', & y \in (-1, 0), \\ G_2(y) &= \int_{-a}^{+a} \exp\left[\frac{-z}{vy}(a - x')\right] g(x', y)\, dx', & y \in (0, 1). \end{aligned}$$

To solve system (12a)-(12b), we introduce the Banach spaces

$$Y_1 = L^1(-1, 0), \quad \|\varphi_1\|_1 = \int_{-1}^{0} |\varphi_1(y)|\, dy; \quad Y_2 = L^1(0, 1), \quad \|\varphi_2\|_2 = \int_{0}^{1} |\varphi_2(y)|\, dy;$$

$$Y = Y_1 \times Y_2, \qquad \|\varphi\|_{12} = \left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\|_{12} = \|\varphi_1\|_1 + \|\varphi_2\|_2,$$

and the operators

$$(14a)\quad B_{21}\varphi_1 = \int_{-1}^{0} h(y, y') \exp(2az/vy')\varphi_1(y')\, dy', \qquad D(B_{21}) = Y_1, \quad R(B_{21}) \subset Y_2,$$

$$(14b)\quad B_{12}\varphi_2 = \int_{0}^{1} k(y, y') \exp(-2az/vy')\varphi_2(y')\, dy', \quad D(B_{12}) = Y_2, \quad R(B_{12}) \subset Y_1,$$

$$(15)\qquad B = \begin{pmatrix} 0, & B_{12} \\ B_{21}, & 0 \end{pmatrix}, \qquad D(B) = Y, \qquad R(B) \subset Y.$$

Then, system (12a)-(12b) becomes

$$(16)\qquad C = BC + F, \qquad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

with

$$(17)\qquad \begin{aligned} F_1(y) &= \int_{0}^{1} k(y, y')\, G_2(y')\, dy', & y \in (-1, 0), \\ F_2(y) &= \int_{-1}^{0} h(y, y')\, G_1(y')\, dy', & y \in (0, 1). \end{aligned}$$

Now, for each $z > 0$, we have from (14) and from (15)

$$\|B_{21}\varphi_1\|_2 \leqslant b \exp(-2az/v)\|\varphi_1\|_1 , \qquad \|B_{12}\varphi_2\|_1 \leqslant b \exp(-2az/v)\|\varphi_2\|_2 ,$$

$$\|B\varphi\|_{12} = \|B_{12}\varphi_2\|_1 + \|B_{21}\varphi_1\|_2 \leqslant b \exp(-2az/v)\|\varphi\|_{12}$$

because of (4). Moreover, we obtain from the two (17)

$$\|F_1\|_1 \leqslant \int_{-1}^{0} dy \int_{0}^{1} k(y, y')|G_2(y')|dy' \leqslant b \int_{0}^{1} |G_2(y')|dy' \leqslant b \int_{0}^{1} dy' \int_{-a}^{+a} |g(x', y')|dx' ,$$

$$\|F_2\|_2 \leqslant b \int_{-1}^{0} dy' \int_{-a}^{+a} |g(x', y')|dx'$$

because of (4) and (13), and so $F \in Y$ with $\|F\|_{12} \leqslant b\|g\| < \infty$, $\forall g \in X$. Since

$$[b \exp(-2az/v)] < 1 \qquad \forall z > 0 , \qquad (I-B)^{-1} \in \mathscr{B}(Y) ,$$

with $\|(I-B)^{-1}\varphi\|_{12} \leqslant [1 - b \exp(-2az/v)]^{-1}\|\varphi\|_{12}$ and (16) gives

$$C = (I-B)^{-1}F = \sum_{j=0}^{\infty} B^j F , \tag{18}$$

with $\|C\|_{12} \leqslant [1 - b \exp(-2az/v)]^{-1}\|F\|_{12} \leqslant b[1 - b \exp(-2az/v)]^{-1}\|g\|$. Relation (18) shows that C_1 and C_2 are uniquely determined by g and depend linearly on g. In other words

$$C_1 = \chi_1 g , \qquad C_2 = \chi_2 g , \tag{19}$$

with $\chi_j \in \mathscr{B}(X, Y_j)$, $j = 1, 2$, because

$$\|C_j\|_j = \|\chi_j g\|_j \leqslant \|C\|_{12} \leqslant b[1 - b \exp(-2az/v)]^{-1}\|g\| . \tag{20}$$

Going back to (11b) ,we have for a.e. $y \in (0, 1)$

$$\int_{-a}^{+a} |f(x, y)|dx \leqslant z^{-1}[1 - \exp(-2az/vy)]|C_2(y)|$$

$$+ (1/vy)\int_{-a}^{+a} dx' \int_{x'}^{+a} \exp[-z(x-x')/vy]|g(x', y)|dx \leqslant z^{-1}|C_2(y)|$$

$$+ z^{-1}\int_{-a}^{+a} \{1 - \exp[-z(a-x')/vy]\}|g(x', y)|dx'$$

$$\leqslant z^{-1}\{|C_2(y)| + \int_{-a}^{+a} |g(x', y)|dx'\} ,$$

$$\int_0^1 \mathrm{d}y \int_{-a}^{+a} |f(x, y)|\,\mathrm{d}x \leqslant z^{-1}\{\|C_2\|_2 + \int_0^1 \mathrm{d}y \int_{-a}^{+a} |g(x', y)|\,\mathrm{d}x'\}\,.$$

Since in an analogous way we obtain from (11a)

$$\int_{-1}^0 \mathrm{d}y \int_{-a}^{+a} |f(x, y)|\,\mathrm{d}x \leqslant z^{-1}\{\|C_1\|_1 + \int_0^1 \mathrm{d}y \int_{-a}^{+a} |g(x', y)|\,\mathrm{d}x'\}\,,$$

we conclude that

$$\text{(21)}\qquad \|f\| = \|(zI - A_0)^{-1}g\| \leqslant z^{-1}\{\|C\|_{12} + \|g\|\}$$

$$\leqslant z^{-1}\{b[1 - b\exp(-2az/v)]^{-1} + 1\}\|g\| \qquad \forall z > 0\,, \quad g \in X\,,$$

and so $(zI - A_0)^{-1} \in \mathscr{B}(X)$, $\forall z > 0$.

(b) If $z > 0$ and $g \in X_0$, then F_1 and F_2 are both nonnegative functions (see (13) and (17)), and (14a), (14b) show that $B_{21}F_1$ and $B_{12}F_2$ are nonnegative. It follows that C_1 and C_2 are nonnegative because of (15) and (18). Thus, $f(x, y)$ is nonnegative (see the two (11)), i.e., $f = (zI - A_0)^{-1}g \in X_0$.

Lemma 3. $\|(zI - A_0)^{-1}g\| \leqslant \|g\|/z$, $\forall z > 0$, $g \in X$.

Proof. If $z > 0$ and $f \in D(A_0) \cap X_0$, we have

$$\|(zI - A_0)f\| \geqslant \Big| \int_{-1}^{+1} \mathrm{d}y \int_{-a}^{+a} [(zI - A_0)f]\,\mathrm{d}x \Big|$$

$$= \Big| z\|f\| + v\int_{-1}^{+1} \mathrm{d}y\,[y\int_{-a}^{+a} (\partial f/\partial x)\,\mathrm{d}x] \Big| = \Big| z\|f\| + v\int_{-1}^{+1} [yf(a, y) - yf(-a, y)]\,\mathrm{d}y \Big|$$

$$= \Big| z\|f\| + v\{\int_0^1 yf(a, y)\,\mathrm{d}y - \int_{-1}^0 |y|f(a, y)\,\mathrm{d}y\} + v\{\int_{-1}^0 |y|f(-a, y)\,\mathrm{d}y - \int_0^1 yf(-a, y)\,\mathrm{d}y\} \Big|\,.$$

But

$$\int_0^1 yf(a, y)\,\mathrm{d}y - \int_{-1}^0 |y|f(a, y)\,\mathrm{d}y$$

$$= \int_0^1 yf(a, y)\,\mathrm{d}y - \int_{-1}^0 \mathrm{d}y \int_0^1 k(y, y')y'f(a, y')\,\mathrm{d}y' = \int_0^1 [1 - \bar{k}(y)]yf(a, y)\,\mathrm{d}y \geqslant 0\,,$$

$$\int_{-1}^0 |y|f(-a, y)\,\mathrm{d}y - \int_0^1 yf(-a, y)\,\mathrm{d}y = \int_{-1}^0 [1 - \bar{h}(y)]|y|f(-a, y)\,\mathrm{d}y \geqslant 0\,,$$

because of (2) and (4) and because $f(x, y) \geqslant 0$ for a.e. $(x, y) \in (-a, a) \times (-1, 1)$.

Hence,

(22) $$\|(zI - A_0)f\| \geqslant z\|f\| \qquad \forall z > 0\,, \quad f \in D(A_0) \cap X_0\,.$$

Now, if $g \in X_0$ and $f = (zI - A_0)^{-1}g$, then $f \in X_0$ because of (b) of Lemma 2, $(zI - A_0)f = g$, and (22) gives

(23) $$\|(zI - A_0)^{-1}g\| \leqslant \|g\|/z \qquad \forall z > 0\,, \quad g \in X_0\,.$$

Finally, if $g \in X$, let $g^-(x, y) = -g(x, y)$ if $g(x, y) < 0$, $g^-(x, y) = 0$ if $g(x, y) \geqslant 0$, $g^+(x, y) = g(x, y)$ if $g(x, y) \geqslant 0$, $g^+(x, y) = 0$ if $g(x, y) < 0$. Then, $g = g^+ - g^-$, $g^+ \in X_0$, $g^- \in X_0$, $\|g\| = \|g^+\| + \|g^-\|$ and we obtain from (23) (see also Appendix A)

$$\|(zI - A_0)^{-1}g\| \leqslant \|(zI - A_0)^{-1}g^+\| + \|(zI - A_0)^{-1}g^-\| \leqslant \big(\|g^+\| + \|g^-\|\big)/z = \|g\|/z\,.$$

Remark 2. If the conditions (4′) are satisfied, then (22) becomes $\|(zI - A_0)f\| = z\|f\|$, $\forall z > 0$, $f \in D(A_0) \cap X_0$, and so

(23)′ $$\|(zI - A_0)^{-1}g\| = \|g\|/z \qquad \forall z > 0\,, \quad g \in X_0\,.$$

3. - The semigroups generated by A_0, A, $A + v\sigma_s J$

The operator A_0 is densely defined because $D(A_0) \supset C_0^\infty\big((-a, a) \times (-1, 1)\big)$. Furthermore, A_0 is closed because $-(zI - A_0)^{-1} \in \mathscr{B}(X) \subset \mathscr{C}(X)$ for each $z > 0$, and so $(A_0 - zI) \in \mathscr{C}(X)$ and $A_0 = (A_0 - zI) + zI \in \mathscr{C}(X)$ because $zI \in \mathscr{B}(X)$. Since A_0 is densely defined and closed, it follows from Lemma 3 that $A_0 \in \mathscr{G}(1, 0; X)$, [**1**], [**4**]. Hence, A_0 generates the strongly continuous semigroup $\{Z_0(t), t \geqslant 0\}$, with $\|Z_0(t)g\| \leqslant \|g\|$, $\forall t \geqslant 0$, and with

$$\lim_{n\to\infty} \|Z_0(t)g - \big(I - (t/n)A_0\big)^{-n}g\| = 0 \qquad \forall t \geqslant 0\,, \quad g \in X\,.$$

Now, if $t > 0$ and $g \in X_0$, then

$$(I - \frac{t}{n}A_0)^{-n}g = [\frac{n}{t}\,(\frac{n}{t}\,I - A_0)^{-1}]^n\, g \in X_0$$

because of (b) of Lemma 2 and so $Z_0(t)g \in X_0$. If $t = 0$ and $g \in X_0$, then $Z_0(g) = g$ obviously belongs to X_0. Thus, we have

Theorem 1. (a) $A_0 \in \mathscr{G}(1, 0; X)$; (b) *the semigroup* $\{Z_0(t), t \geqslant 0\}$ *generated by* A_0 *maps* X_0 *into itself for any* $t \geqslant 0$.

Remark 3. If the conditions (4)′ are satisfied, then (23′) gives

$$\|(I - \frac{t}{n} A_0)^{-1} g\| = \frac{n}{t} \|(\frac{n}{t} I - A_0)^{-1} g\| = \|g\| \qquad \forall g \in X_0 ,$$

$n = 1, 2, \ldots, t > 0$. Hence, $\|[(I - (t/n) A_0)^{-1}]^n g\| = \|g\|$, and so

$$\|Z_0(t) g\| = \|g\| \qquad \forall t \geqslant 0 , \qquad g \in X_0 . \tag{24}$$

The physical meaning of relation (24) will be discussed later on, (see (28), (29) and the discussion that follows).

Theorem 2. (a) *The operator* A *generates the semigroup* $\{Z(t), t \geqslant 0\}$, *which is such that* $Z(t) = \exp(-v\sigma t) Z_0(t)$, $\forall t \geqslant 0$; (b) $A + v\sigma_s J \in \mathscr{G}(1, v(\sigma_s - \sigma); X)$; (c) *if* $\{S(t), t \geqslant 0\}$ *is the semigroup generated by* $A + v\sigma_s J$, *then* $S(t) g \in X_0$, $\forall g \in X_0$, $t \geqslant 0$.

Proof. (a) is obvious because $-v\sigma I$ commutes with A_0. (b) $A \in \mathscr{G}(1, -v\sigma; X)$ because $\|Z(t) f\| \leqslant \exp(-v\sigma t)\|f\|$, and $v\sigma_s J \in \mathscr{B}(X)$ with $\|v\sigma_s J f\| \leqslant v\sigma_s \|f\|$ because of (a) of Lemma 1. Hence, $A + v\sigma_s J \in \mathscr{G}(1, -v\sigma + v\sigma_s; X)$. (c) We have

$$\lim_{n \to \infty} \|S(t) g - \sum_{j=0}^{n} Z_j(t) g\| = 0 , \qquad t \geqslant 0, \qquad g \in X ,$$

with

$$Z_0(t) g = Z(t) g , \qquad Z_{j+1}(t) g = v\sigma_s \int_0^t Z(t-s) J Z_j(s) g \, ds , \qquad j = 0, 1, \ldots .$$

Thus, if $g \in X_0$, $Z_0(t) g = \exp(-v\sigma t) Z_0(t) g$, $Z_1(t) g$, $Z_2(t) g$, ..., all belong to X_0 because of (b) of Theorem 1. If follows that $S(t) g$ also belongs to the closed cone X_0.

Remark 4. Under the assumptions (4′), (24) and (a) of Theorem 2 give: $\|Z_0(t) g\| = \|Z(t) g\| = \exp(-v\sigma t)\|g\|$, $g \in X_0$, $t \geqslant 0$, and so

$$\begin{aligned} \|Z_1(t) g\| &= v\sigma_s \| \int_0^t Z(t-s) J Z_0(s) g \, ds \| \\ &= v\sigma_s \int_0^t \|Z(t-s) J Z_0(s) g\| ds = v\sigma_s \int_0^t \exp[-v\sigma(t-s)] \|J Z_0(s) g\| ds \\ &= v\sigma_s \int_0^t \exp[-v\sigma(t-s)] \|Z_0(s) g\| ds = v\sigma_s t \exp(-v\sigma t) \|g\| \end{aligned}$$

because $Z(t-s)JZ_0(s)g \in X_0$ $\forall s \in [0, t]$. By a similar procedure, we have $\|Z_j(t)g\| = [(v\sigma_s t)^j/j!]\exp(-v\sigma t)\|g\|$, $\forall g \in X_0$, $t \geq 0$, and so

$$\text{(25)} \qquad \|S(t)g\| = \sum_{j=0}^{\infty} [(v\sigma_s t)^j/j!]\exp(-v\sigma t)\|g\|$$
$$= \exp[v(\sigma_s - \sigma)t]\,\|g\| \qquad \forall g \in X_0, \quad t \geq 0 .$$

4. - The abstract problem (9)

If $u_0 \in D(A) \cap X_0$ and $q = q(t)$ is continuously differentiable and belongs to X_0 at any $t \geq 0$, then the unique strict solution of (9) can be written as follows

$$\text{(26)} \qquad u(t) = S(t)u_0 + \int_0^t S(t-s)q(s)\,\mathrm{d}s , \qquad t \geq 0 ,$$

and $u(t) \in D(A) \cap X_0$ $\forall t \geq 0$ because of (c) of Theorem 2, [1], [4]. We have from (26)

$$\text{(27)} \qquad \|u(t)\| = \|S(t)u_0\| + \int_0^t \|S(t-s)q(s)\|\,\mathrm{d}s$$
$$\leq \exp[v(\sigma_s - \sigma)t]\|u_0\| + \int_0^t \exp[v(\sigma_s - \sigma)(t-s)]\|q(s)\|\,\mathrm{d}s ,$$

where $\|u(t)\| = N(t)$, $\|u_0\| = N(0)$ are the total numbers of particles in the slab at time t and at time $t = 0$, see (5).

If in particular assumptions (4)′ are satisfied, then (25) and (26) give

$$\text{(28)} \qquad \|u(t)\| = \exp[v(\sigma_s - \sigma)t]\|u_0\| + \int_0^t \exp[v(\sigma_s - \sigma)(t-s)]\|q(s)\|\,\mathrm{d}s ,$$

and so

$$\text{(29a)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} N(t) = v(\sigma_s - \sigma)N(t) + \|q(t)\| , \qquad \text{(29b)} \quad N(0) = \|u_0\| .$$

(29a) is an equation of balance which takes into account that particles are not captured by the boundary planes. Note that (29a) can be derived in a *heuristic* way by integrating both sides of (1) with respect to x and y and taking into account (2a), (2b), and (4′). However, as it was proved above, (29a) can be obtained by a *rigorous* procedure from (26) and by using (24), which shows that the evolution operator $Z_0(t)$ generated by the free-streaming operator A_0 does not change the total number of particles in the slab (provided that (4′) are satisfied).

The above results can be summarized as follows.

Theorem 3. *Assume that* $u_0 \in D(A) \cap X_0$ *and that* $q(t)$ *is continuously differentiable and belongs to* X_0 *at any* $t \geq 0$. *Then if the assumptions* (4) *are satisfied the evolution problem* (9) *has a unique strict solution* $u = u(t)$ *that is defined by* (26), *belongs to* $D(A) \cap X_0$, *and satisfies the inequality* (27). *Moreover, under the assumptions* (4′), $N(t) = \|u(t)\|$ *is the solution of system* (29a)-(29b).

Appendixes

A. – The properties $(zI - A_0)^{-1} \in \mathscr{B}(X)$ and $\|(zI - A_0)^{-1}\| \leq \|g\|/z \quad \forall z > 0$, $g \in X$ can be derived directly from (11a)-(11b) as follows.

If $y \in (-1, 0)$, then we have from (11a)

$$\int_{-a}^{+a} |f(x, y)|\,\mathrm{d}x \leq z^{-1}[1 - \exp(2za/vy)]\,|C_1(y)| + z^{-1}\int_{-a}^{+a}\{1 - \exp[z(a + x')/vy]\}\,|g(x', y)|\,\mathrm{d}x',$$

whereas (11b) gives for $y \in (0, 1)$

$$\int_{-a}^{+a} |f(x, y)|\,\mathrm{d}x \leq z^{-1}[1 - \exp(-2za/vy)]\,|C_2(y)| + z^{-1}\int_{-a}^{+a}\{1 - \exp[-z(a - x')/vy]\}\,|g(x', y)|\,\mathrm{d}x'.$$

On the other hand, we obtain from (12a) and from (12b)

$$\int_{-1}^{0} |C_1(y)|\,\mathrm{d}y \leq \int_{0}^{1} \bar{k}(y')\exp(-2az/vy')\,|C_2(y')|\,\mathrm{d}y' + \int_{0}^{1}[\bar{k}(y')\int_{-a}^{+a}\exp[-z(a - x')/vy']\,|g(x', y')|\,\mathrm{d}x']\,\mathrm{d}y',$$

$$\int_{0}^{1} |C_2(y)|\,\mathrm{d}y \leq \int_{-1}^{0} \bar{h}(y')\exp(2az/vy')\,|C_1(y')|\,\mathrm{d}y' + \int_{-1}^{0}[\bar{h}(y')\int_{-a}^{+a}\exp[z(a + x')/vy']\,|g(x', y')|\,\mathrm{d}x']\,\mathrm{d}y'$$

and so

$$z\|f\| \leqslant \int_{-1}^{0} |C_1(y)|\,\mathrm{d}y - \int_{-1}^{0} \exp(2za/vy')\,|C_1(y')|\,\mathrm{d}y'$$

$$+ \int_{0}^{1} |C_2(y)|\,\mathrm{d}y - \int_{0}^{1} \exp(-2za/vy')\,|C_2(y')|\,\mathrm{d}y'$$

$$+ \int_{-1}^{0} \mathrm{d}y' \int_{-a}^{+a} \{1 - \exp[z(a + x')/vy']\}\,|g(x', y')|\,\mathrm{d}x'$$

$$+ \int_{0}^{1} \mathrm{d}y' \int_{-a}^{+a} \{1 - \exp[-z(a - x')/vy']\}\,|g(x', y')|\,\mathrm{d}x'$$

$$\leqslant \int_{0}^{1} [\bar{k}(y') - 1]\exp(-2za/vy')\,|C_2(y')|\,\mathrm{d}y'$$

$$+ \int_{-1}^{0} [\bar{h}(y') - 1]\exp(2za/vy')\,|C_1(y')|\,\mathrm{d}y'$$

$$+ \int_{0}^{1} \mathrm{d}y'[\bar{k}(y') - 1] \int_{-a}^{+a} \exp[-z(a - x')/vy']\,|g(x', y')|\,|\,\mathrm{d}x'$$

$$+ \int_{-1}^{0} \mathrm{d}y'[\bar{h}(y') - 1] \int_{-a}^{+a} \exp[z(a + x')/vy']\,|g(x', y')|\,\mathrm{d}x'$$

$$+ \int_{-1}^{+1} \mathrm{d}y' \int_{-a}^{+a} |g(x', y')|\,\mathrm{d}x'\,.$$

Hence, if $z > 0$ and $g \in X$, we have

$$z\|f\| \leqslant \|g\| + (b - 1)\exp(-2za/v)\|C\|_{12} + (b - 1)\|g\| \tag{30}$$

$$\leqslant b\|g\| + \frac{b(b-1)\exp(-2za/v)}{1 - b\exp(-2za/v)}\|g\| = b\,\frac{1 - \exp(-2za/v)}{1 - b\exp(-2za/v)}\|g\|,$$

and so

$$z\|f\| \leqslant \|g\| \tag{31}$$

provided that $b \leqslant 1$.

Remark 5. If $\bar{k}(y') = 1$ $\forall y' \in (0, 1)$, $\bar{h}(y') = 1$ $\forall y' \in (-1, 0)$, (see (4′)), and if $g \in X_0$, then the above procedures give $z\|f\| = \|g\|$.

Remark 6. Inequality (30) holds even if $b > 1$. However, in this case, z must be taken larger than z_0, with $[b \exp(-2z_0 a/v)] = 1$. In fact, if $z > z_0$, then $[b \exp(-2za/v)] < 1$ and $(I - B)^{-1} \in \mathscr{B}(Y)$, i.e., C is uniquely determined by g, (see the proof of (a) of Lemma 2).

B. – Assume that (i) $\sigma = \sigma_s$; (ii) $h(y, y') = 2y$ for a.e. $(y, y') \in (0, 1) \times (-1, 0)$, $k(y, y') = -2y$ for a.e. $(y, y') \in (-1, 0) \times (0, 1)$. Note that (i) and (ii) imply that the materials of the slab under consideration and the boundary planes do not capture particles. Assumption (ii) also shows that an outgoing isotropic density is reflected isotropically (i.e., if for instance $f(-a, y') = c = a$ *constant* for a.e. $y' \in (-1, 0)$, then $f(-a, y) = c$ for a.e. $y \in (0, 1)$).

Under the assumptions (i), (ii), it is not difficult to show that the equation

$$(A_0 - v\sigma I + v\sigma J) f = 0 \tag{32}$$

has the solution $f(x, y) = c$. In other words, (i) and (ii) imply that the transport operator $(A_0 - v\sigma I + v\sigma J)$ has the eigenvalue $z = 0$ and that the corresponding eigenfunction f does not depend on x and y. Thus, if $u_0(x, y) = c$ and $q(t) = 0$ $\forall t \geqslant 0$, then the unique solution of (9) is $u(t) = u(x, y; t) = c$, $\forall t \geqslant 0$.

To show that $f(x, y) = c$ satisfies (32), we re-write (32) into the *equivalent* form

$$f = v\sigma(v\sigma I - A_0)^{-1} J f, \tag{33}$$

where $v\sigma(v\sigma I - A_0)^{-1} J f$ is given by (11a)-(11b) with $z = v\sigma$ and with $g = v\sigma J f$. Now, if $f(x, y) = c$, then $g = v\sigma c$ and (11a)-(11b) with $z = v\sigma$ give

$$\begin{aligned} c &= (-1/vy)\, C_1(y) \exp[\sigma(a - x)/y] \\ &\qquad + c\{1 - \exp[-\sigma(x - a)/y]\}, \quad y \in (-1, 0), \\ c &= (1/vy)\, C_2(y) \exp[-\sigma(a + x)/y] \\ &\qquad + c\{1 - \exp[-\sigma(x + a)/y]\}, \quad y \in (0, 1). \end{aligned} \tag{34}$$

On the other hand, we have from (12a)-(12b) with $z = v\sigma$

$$C_1(y)/(-y) = 2 \int_0^1 \exp(-2a\sigma/y')\, C_2(y')\, dy' + 2 \int_0^1 G_2(y')\, dy', \qquad y \in (-1, 0),$$

$$C_2(y)/y = 2 \int_{-1}^0 \exp(2a\sigma/y')\, C_1(y')\, dy' + 2 \int_{-1}^0 G_1(y')\, dy', \qquad y \in (0, 1),$$

and so $C_1(y)/(-y) = c_1$, $C_2(y)/y = c_2$. Hence

$$c_1 = \quad 2c_2 \int_0^1 y' \exp(-2a\sigma/y')\,dy' + 2vc \int_0^1 y'[1 - \exp(-2a\sigma/y')]\,dy',$$

$$c_2 = -2c_1 \int_{-1}^0 y' \exp(2a\sigma/y')\,dy' + 2vc \int_{-1}^0 y'[\exp(2a\sigma/y') - 1]\,dy',$$

i.e.,

$$c_1 = 2(c_2 - vc) \int_0^1 y' \exp(-2a\sigma/y')\,dy' + vc,$$

$$c_2 = 2(c_1 - vc) \int_0^1 y' \exp(-2a\sigma/y')\,dy' + vc.$$

It follows that $c_1 = c_2 = vc$, $C_1(y) = -yvc$, $C_2(y) = yvc$, and that consequently the two (34) are identically satisfied. Thus, $f(x, y) = c$ satisfies (33) and (32).

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Sommario

Si prova esistenza ed unicità di una soluzione sommabile per un problema integrodifferenziale della teoria del trasporto di particelle in un muro omogeneo limitato da piani capaci di catturare e di riflettere particelle. Si studiano quindi alcune proprietà di tale soluzione.

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