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Existence theorem for almost periodic solutions of periodic systems with time lag (**)

Let $\tau > 0$ be some real number, and let $\mathscr{C}$ denote the Banach space of continuous functions mapping the compact interval $[-\tau, 0]$ into $\boldsymbol{R}^n$, with the norm of uniform convergence, i.e. the norm of $\varphi \in \mathscr{C}$ is given by $\|\varphi\|_* = \sup_{-\tau \leqslant \vartheta \leqslant 0} \|\varphi(\theta)\|$, where $\|\cdot\|$ is a norm in $\boldsymbol{R}^n$. If $x\colon [t_0 - \tau, \infty) \to \boldsymbol{R}^n$ is a continuous function, then for any $t \geqslant t_0$ let the symbol x_t denote the function $x_t\colon [-\tau, 0] \to \boldsymbol{R}^n$, defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leqslant \theta \leqslant 0$. For every $t \geqslant t_0$, x_t is an element of $\mathscr{C}$ (t has to be considered as a parameter). Finally let $\varrho > 0$ be a constant, and let $\mathscr{C}_\varrho$ denote the set of $\varphi \in \mathscr{C}$ such that $\|\varphi\|_* < \varrho$, and consider the system of functional differential equations (of retarded type)

$$\dot{x}(t) = f(t, x_t) \tag{1}$$

where the functional $f\colon \boldsymbol{R} \times \mathscr{C}_\varrho \to \boldsymbol{R}^n$ satisfies the following conditions

(i) f is continuous on $\boldsymbol{R} \times \mathscr{C}_\varrho$ and takes closed bounded sets of $\boldsymbol{R} \times \mathscr{C}_\varrho$ into bounded sets of $\boldsymbol{R}^n$,

(ii) f is periodic with respect to t, i.e. for some period $T > 0$ there is $f(t + T, \varphi) \equiv f(t, \varphi)$,

(iii) f is smooth enough to ensure uniqueness for initial value problems and continuous dependence on initial conditions of the solutions of the system (1). Then the maximal solution of (1) through $(t_0, \varphi_0) \in \boldsymbol{R} \times \mathscr{C}_\varrho$ will be denoted by $x(t_0, \varphi_0)$.

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Let us suppose that (1) possesses a solution $x(t)$ *bounded in the future*, i.e. defined on $[t_0 - \tau, \infty)$ and such that $\|x(t)\| \leqslant L < \varrho$, $\forall t \geqslant t_0 - \tau$. Owing to (i) and (ii), there is a positive constant M such that $\|f(t, \varphi)\| \leqslant M$, $\forall t \in \boldsymbol{R}$, $\forall \varphi \in \mathscr{C}$ with $\|\varphi\|_* \leqslant L$, and therefore $\|\dot{x}(t)\| \leqslant M$, $\forall t \in (t_0, \infty)$, from which follows

$$\|x(t_1) - x(t_2)\| \leqslant M\,|t_1 - t_2|\,, \qquad \forall t_1, t_2 \in [t_0, \infty)\,. \tag{2}$$

We extend now the concept of T-stability, introduced in [**3**]$_1$, to the solutions of T-periodic systems of functional differential equations as follows

Definition. A solution $x\colon [t_0 - \tau, \infty) \to \boldsymbol{R}^n$ of (1) is called *T-stable*, if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for any two positive integers n, m with $\sup\limits_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta + nT) - x(t_0 + \theta + mT)\| < \delta$, it follows that $\|x(t + nT) - x(t + mT)\| < \varepsilon$ for all $t \geqslant t_0$, i.e. using the logical quantifiers

$$\begin{gathered}(\forall \varepsilon > 0)\ (\exists \delta > 0)\ \big(\forall n, m \in \boldsymbol{N}\colon \sup_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta + nT) - x(t_0 + \theta + mT)\| < \delta\big)\ (\forall t \geqslant t_0) \\ \|x(t + nT) - x(t + mT)\| < \varepsilon\,,\end{gathered} \tag{3}$$

where $\boldsymbol{N}$ is the set of positive integers.

An equivalent condition of T-stability is given by the following

Lemma. *A solution $x\colon [t_0 - \tau, \infty) \to \boldsymbol{R}^n$ of (1) is T-stable if and only if it is*

$$\begin{gathered}(\forall \varepsilon > 0)\ (\exists \delta > 0)\ \big(\forall m \in \boldsymbol{N}\colon \sup_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta) - x(t_0 + \theta + mT)\| < \delta\big)\ (\forall t \geqslant t_0) \\ \|x(t) - x(t + mT)\| < \varepsilon\,.\end{gathered} \tag{4}$$

In fact (4) follows from (3), putting $n = 0$. We prove now that condition (4) implies T-stability with the same pair $(\varepsilon, \delta(\varepsilon))$. Let $\bar{n}, \bar{m} \in \boldsymbol{N}$ $(\bar{n} \geqslant \bar{m})$ be such that $\sup\limits_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta + \bar{n}T) - x(t_0 + \theta + \bar{m}T)\| < \delta$. With the change of variable: $t \to t + \bar{m}T$ and for the particular value $m = \bar{n} - \bar{m} \geqslant 0$, we obtain from (4) the following relation: $\sup\limits_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta + \bar{m}T) - x(t_0 + \theta + \bar{n}T)\| < \delta \Rightarrow$ $\Rightarrow \|x(t + \bar{m}T) - x(t + \bar{n}T)\| < \varepsilon\,,\ \ \forall t \geqslant t_0\,,$ which proves the Lemma.

We are now able to prove the following

Theorem. *Let* $x\colon [t_0 - \tau, \infty) \to \mathbf{R}^n$ *be a solution bounded in the future of the functional differential system* (1), *satisfying the above-mentioned conditions* (i), (ii), (iii). *Then the solution* $x(t)$ *is asymptotically almost periodic if and only if* $x(t)$ *is* T*-stable.*

The condition is sufficient. Let $\{\alpha_n\}$ be any sequence of real numbers with $\alpha_n \to \infty$ for $n \to \infty$. The existence of a subsequence $\{\alpha'_n\}$ of $\{\alpha_n\}$, such that the sequence of functions $\{x(t + \alpha'_n)\}$ is uniformly convergent for $t \in [t_0, \infty)$, assures that $x(t)$ is asymptotically almost periodic (like Fréchet [1]). For every α_n there is an (unique) integer k_n such that $k_n T \leqslant \alpha_n < (k_n + 1)T$, where $k_n \geqslant \tau/T$ if n is sufficiently large. Put $\beta_n = \alpha_n - k_n T$. The sequence of functions $\{x(t_0 + \theta + k_n T)\}$, $-\tau \leqslant \theta \leqslant 0$, is *equicontinuous* (due to (2)) and *equibounded* (because the solution $x(t)$ is bounded in the future), therefore there is a subsequence $\{k'_n\}$ of $\{k_n\}$, such that $\{x(t_0 + \theta + k'_n T)\}$ converges uniformly for $\theta \in [-\tau, 0]$. Moreover the real sequence $\{\beta_n\}$ is bounded and therefore there exists a convergent subsequence $\{\beta'_n\}$. Let $\{\beta'_n\}$ and $\{k'_n\}$ be two common subsequences of $\{\beta_n\}$ and $\{k_n\}$ respectively. Then $\{\alpha'_n\}$, where $\alpha'_n = \beta'_n + k'_n T$, is a subsequence of $\{\alpha_n\}$, and we shall prove that the sequence $\{x(t + \alpha'_n)\}$ is uniformly convergent on $[t_0, \infty)$. In fact, let $\varepsilon > 0$ be given. Then let us denote by $\delta_1 = \delta_1(\varepsilon) > 0$ a value of δ corresponding to $\varepsilon/3$ in the condition (3) of T-stability, and put $\delta = \min\{\varepsilon/3M, \delta_1(\varepsilon)\}$. There is a positive integer $K = K(\varepsilon)$ such that for all $n, m \geqslant K$ it is: $|\beta'_n - \beta'_m| < \delta$ and $\sup\limits_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta + k'_n T) - x(t_0 + \theta + k'_m T)\| < \delta$. Therefore it follows that: $\|x(t + \beta'_n) - x(t + \beta'_m)\| \leqslant M|\beta'_n - \beta'_m| < \varepsilon/3$ for all $t \geqslant t_0$ (owing to (2)), and $\|x(t + k'_n T) - x(t + k'_m T)\| < \varepsilon/3$ for all $t \geqslant t_0$ (due to the T-stability). Then

$$\|x(t + \alpha'_n) - x(t + \alpha'_m)\| \leqslant \|x(t + \beta'_n + k'_n T) - x(t + \beta'_K + k'_n T)\| + \\ + \|x(t + \beta'_K + k'_n T) - x(t + \beta'_K + k'_m T)\| + \|x(t + \beta'_K + k'_m T) - x(t + \beta'_m + k'_m T)\| < \varepsilon ,$$

for all integers $n, m \geqslant K$ and all $t \geqslant t_0$, which proves that $\{x(t + \alpha'_n)\}$ is uniformly convergent on $[t_0, \infty)$ and therefore $x(t)$ is asymptotically almost periodic.

The condition is necessary. Let $\varepsilon > 0$ be given. Because $x(t)$ is asymptotically almost periodic, there is a $T_0 = T_0(\varepsilon) > t_0$ and an $l = l(\varepsilon) > 0$ such that for any $t > T_0 + l$ there exists a $t' \in [T_0, T_0 + l]$ such that

$$\|x(t + mT) - x(t' + mT)\| < \varepsilon/3 , \qquad \forall m \in \mathbf{N}, \tag{5}$$

(see footnote (1) of [3]$_2$, p. 56). Moreover, due to the hypothesis of continuous dependence on initial conditions, there is a $\delta = \delta(\varepsilon) > 0$ (δ depends only on ε, because t_0 has to be considered as fixed) such that on the interval $[t_0, T_0 + l]$ it is

$$(6)\qquad \big(\forall m \in \boldsymbol{N}\colon \sup_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta) - x(t_0 + \theta + mT)\| < \delta\big) \quad (\forall t' \in [t_0, T_0 + l]) \\ \|x(t') - x(t' + mT)\| < \varepsilon/3 \,.$$

In correspondence to this value of $\delta = \delta(\varepsilon)$, the condition (4) of the Lemma is satisfied. In fact, for any $t > T_0 + l$ there exists a $t' \in [T_0, T_0 + l]$ such that (5) holds and furthermore by (6), for any positive integer m such that $\sup\limits_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta) - x(t_0 + \theta + mT)\| < \delta$, it follows that $\|x(t') - x(t' + mT)\| < \varepsilon/3$. Therefore, it is

$$\|x(t) - x(t + mT)\| \leqslant \|x(t) - x(t')\| + \\ + \|x(t') - x(t' + mT)\| + \|x(t' + mT) - x(t + mT)\| < \varepsilon \,,$$

and by the Lemma the solution $x(t)$ is T-stable.

Remark. If in (4) we choose a δ such that $0 < \delta < \varepsilon$, the condition (4) of the Lemma can be put in the following equivalent form

$$(7)\qquad (\forall \varepsilon > 0) \quad (\exists \delta > 0) \quad \big(\forall m \in \boldsymbol{N}\colon \sup_{-\tau \leqslant \theta \leqslant 0} \|x(t_0 + \theta) - x(t_0 + \theta + mT)\| < \delta\big) \quad (\forall t \geqslant t_0) \\ \sup_{-\tau \leqslant \theta \leqslant 0} \|x(t + \theta) - x(t + \theta + mT)\| < \varepsilon \,,$$

which shows that T-stability is a *conditional stability*, i.e. the solution $x(t)$ is stable, at t_0, *only* with respect to the set of translates $x(t + mT)$, $m = 1, 2, \ldots$. This type of stability is of course weaker than Liapunov stability, defined as follows: the solution $x(t)$ of (1) is called stable at t_0, if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for *any* solution $x(t_0, \varphi_0)$ of (1) with $\|\varphi_0 - x_{t_0}\|_* < \delta$, it follows that $\|x_t - x_t(t_0, \varphi_0)\|_* < \varepsilon$ for all $t \geqslant t_0$. Therefore the sufficient condition of our Theorem is a generalization of Halanay well-known theorem ([**2**], p. 486).

References

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Abstract

The present paper extends to functional differential equations some results established in [**3**]$_1$, [**3**]$_2$ *for ordinary differential equations. The main result is a necessary and sufficient condition for solutions bounded in the future of periodic systems with time lag to be asymptotically almost periodic (which in turn assures the existence of almost periodic solutions).*

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