Riv. Mat. Univ. Parma (4) 4 (1978), 385-393

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Absolute summability by (E, q) means (**)

1. - Definitions and notations.

Let $\sum a_n$ (1) be a given infinite series. Then the series $\sum a_n$ is said to be absolutely summable (E, q) $(q \geqslant 0)$ or symbolically we write $\sum a_n \in |E, q|$ $(q \geqslant 0)$, if

(1.1) $$\sum |A_n^{(q)}| < \infty ,$$

where (see Hardy [2], p. 178 and theorem 185)

(1.2) $$A_n^{(q)} = (q+1)^{-n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} a_m .$$

Let f be 2π-periodic and $\in L(-\pi, \pi)$, and let its Fourier series and conjugate series, at a point $t = x$, be respectively: $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum A_n(x)$ and $\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum B_n(x)$, assuming $B_0(x) = 0$.

We use the following notations:

(1.3) $\varphi(t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right)$, (1.4) $\psi(t) = \frac{1}{2}\left(f(x+t) - f(x-t)\right)$,

(1.5) $\varphi_1(t) = \frac{1}{t} \int_0^t \varphi(u)\,du$, (1.6) $\psi_1(t) = \frac{1}{t} \int_0^t \psi(u)\,du$,

(1.7) $R(t) = \varphi(t) - \varphi_1(t)$, (1.8) $S(t) = \psi(t) - \psi_1(t)$.

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(1) Summations are over 0, 1, 2, ..., ∞, when there is no indication to the contrary.

Let $g(k/t)$ stand for any one of the following functions:

$$\left(\log\frac{k}{t}\right)^{1+c}, \quad \log\frac{k}{t}\left(\log_2\frac{k}{t}\right)^{1+c}, \quad \dots,$$

$$\log\frac{k}{t} - \dots \quad \log_{b-1}\frac{k}{t}\left(\log_b\frac{k}{t}\right)^{1+c},$$

where $c > 0$ and k is some positive constant chosen for the convenience in the analysis and not necessarily the same at each occurrence.

2. - Introduction.

Concerning the absolute (E, q) summability of Fourier series, at a point $t = x$, the following theorems are known.

Theorem A. Let $0 < p < 1$. Then $\varphi(t)\log 1/t \in BV(0, p)$ implies

$$\sum A_n(x) \in |E, q| \qquad (0 < q < 1)$$

(Mohanty and Mohapatra [**3**]).

Theorem B. Let $0 < p < 1$ and $\varphi(t) \in BV(0, p)$. Then, for

$$t\beta(t) = \int_0^t u\,\mathrm{d}\varphi(u)\,, \qquad \beta(t)\,g(k/t) \in BV(0, p)$$

implies $\sum A_n(x) \in |E, q|$ $(0 < q < 1)$ (Chandra $[\mathbf{1}]_1$).

For the conjugate series of the Fourier series at $t = x$, the following theorems are known.

Theorem C. Let $0 < p < 1$ and $\psi(+0) = 0$. Then $\int_0^p \log(1/t)\,|\mathrm{d}\psi(t)| < \infty$ implies $\sum B_n(x) \in |E, q|$ $(0 < q < 1)$ (Mohanty and Mohapatra [**3**]).

Theorem D. Let $0 < p < 1$. Then $\psi(t)\,g(k/t) \in BV(0, p)$ implies

$$\sum B_n(x) \in |E, q| \qquad (0 < q < 1)$$

(Chandra $[\mathbf{1}]_2$).

The object of this paper is to prove the following

Theorem 1. *Let* $0 < p \leqslant e^{-2}$ *and* $(R(t)/t)\log 1/t \in L(0, p)$. *Then*

$$R(t)\log 1/t \in BV(0, p)$$

implies $\sum A_n(x) \in |E, q|$ $(0 < q < 1)$.

Theorem 2. *Let* $0 < p \leqslant e^{-2}$ *and* $(R(t)/t)\log 1/t \in L(0, p)$. *Then*

$$\int_0^p \log(1/t)\ |dR(t)| < \infty$$

implies $\sum A_n(x) \in |E, q|$ $(0 < q < 1)$.

Theorem 3. *Let* $0 < p \leqslant e^{-2}$ *and* $S(+0) = 0$. *Then*

$$\frac{S(t)}{t}\log(1/t) \in L(0, p) \quad \text{and} \quad \int_0^p \log(1/t)\ |dS(t)| < \infty$$

implies $\sum B_n(x) \in |E, q|$ $(0 < q < 1)$.

3. – We shall use the following lemmas in the proof of the theorems.

Lemma 1. *If* $\sum a_n \in |C, 0|$, *then* $\sum a_n \in |E, q|$ $(q > 0)$.

The proof of the lemma, being simple, has been omitted.

Lemma 2. *Let* $0 < p \leqslant e^{-2}$ *and* $F(t) \in L(0, p)$. *Then* $\sum Q_n \in |E, q|$ $(0 < q < 1)$, *where*

$$Q_n = \int_p^\pi F(t)\exp(int)\,dt\,.$$

Proof. $\sum Q_n \in |E, q|$ $(0 < q < 1)$, if, by (1.1) and (1.2),

$$W = \sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m} Q_m\right| < \infty\,.$$

Now

$$W \leqslant \int_p^\pi |F(t)|\,dt \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \exp(imt) \right|$$

$$\leqslant \int_p^\pi |F(t)|\,dt \sum \left(\frac{q}{1+q}\right)^n \left| \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\exp(it)}{q}\right)^m \right|$$

$$\leqslant \int_p^\pi |F(t)|\,dt \sum \left(\frac{q}{1+q}\right)^n \left| \left(\frac{q+\exp(it)}{q}\right)^n \right|$$

$$\leqslant \int_p^\pi |F(t)| \left(\sum (q+1)^{-n} (1+q^2+2q\cos t)^{n/2} |\exp(iny)|\right) dt$$

$$[y = \tan^{-1}(\sin t/(q+\cos t))]$$

$$\leqslant \int_p^\pi |F(t)| \left(\sum \left(1 - \frac{4q}{(1+q)^2} \sin^2 \tfrac{1}{2} t\right)^{n/2}\right) dt$$

$$\leqslant \int_p^\pi |F(t)| \left(\sum (1-\sin^2 \tfrac{1}{2} s)^{n/2}\right) dt \left(\frac{2\sqrt{q}}{1+q} \sin \tfrac{1}{2} t = \sin \tfrac{1}{2} s\right)$$

$$\leqslant \int_p^\pi |F(t)| (1-\cos \tfrac{1}{2} s)^{-1} dt = O\left\{ \int_p^\pi |F(t)| t^{-2} dt \right\} = O(1) .$$

This completes the proof of the lemma.

Lemma 3. *Let* $0 < q < 1$. *Then*

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\exp(imt)}{m+1} \right| = O\left(\log \frac{1}{t}\right),$$

uniformly in $0 < t \leqslant p \leqslant e^{-2}$.

The proof of the lemma runs parallel to the proof of lemma 3 of Chandra [1]$_1$.

4. - Proof of the theorems.

4.1. – Proof of Theorem 1. We have

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \varphi(t) \cos nt\,dt = \frac{2}{\pi} \left(\int_0^p + \int_p^\pi \right) \varphi(t) \cos nt\,dt = \frac{2}{\pi}(P_n + Q_n), \quad \text{say}.$$

The series $\sum (2/\pi) Q_n \in |E, q|$ $(0 < q < 1)$ by Lemma 2. Therefore, for the proof of the theorem, we only require to prove $\sum P_n \in |E, q|$ $(0 < q < 1)$. Now

$$P_n = \int_0^p \varphi(t) \cos nt \, dt = \int_0^p \varphi(t) \left(\cos nt - \frac{\sin nt}{nt} \right) dt + \int_0^p \varphi(t) \frac{\sin nt}{nt} \, dt$$

$$= \int_0^p \varphi(t) \left(\cos nt - \frac{\sin nt}{nt} \right) dt - \int_0^p \varphi(t) \, dt \int_t^\pi \frac{\partial}{\partial u} \left(\frac{\sin nu}{nu} \right) du$$

$$= \varphi_1(p) \frac{\sin np}{n} + \int_0^p \varphi(t) \left(\cos nt - \frac{\sin nt}{nt} \right) dt - \int_0^p \varphi(t) \, dt \int_t^p \frac{\partial}{\partial u} \left(\frac{\sin nu}{nu} \right) du$$

$$= \varphi_1(p) \frac{\sin np}{n} + \int_0^p t R(t) \frac{\partial}{\partial t} \left(\frac{\sin nt}{nt} \right) dt$$

(by changing the order of integration and using (1.5) and (1.7)),

$$P_n = \varphi_1(p) \frac{\sin np}{n} - \int_0^p \frac{\sin nt}{nt} R(t) \, dt + \int_0^p R(t) \cos nt \, dt \tag{4.1.1}$$

$$= \varphi_1(p) \frac{\sin np}{n} - \int_0^p \frac{\sin nt}{nt} R(t) \, dt + R(p) \log \frac{1}{p} \int_0^p \frac{\cos nu}{\log 1/u} \, du$$

$$- \int_0^p d\left(R(t) \log \frac{1}{t} \right) \int_0^t \frac{\cos nu}{\log 1/u} \, du \, .$$

Now, for $0 < t \leqslant p$, we have

$$\int_0^t \frac{\cos nu}{\log 1/u} \, du = \frac{\sin nt}{n \log 1/t} + O \left\{ \frac{1}{(n+1)(\log (n+2))^2} \right\},$$

therefore

$$P_n = \varphi(p) \frac{\sin np}{n} - \int_0^p \frac{\sin nt}{nt} R(t) \, dt + O \left\{ \frac{|R(p)| \log (1/p)}{(n+1)(\log (n+2))^2} \right\}$$

$$- \int_0^p d\left(R(t) \log \frac{1}{t} \right) \frac{\sin nt}{n \log 1/t} + O \left\{ \frac{1}{(n+1)(\log (n+2))^2} \int_0^p \left| d\left(R(t) \log \frac{1}{t} \right) \right| \right\}.$$

The series $\sum P_n \in |E, q|$ $(0 < q < 1)$, if, by (1.1) and (1.2)

$$Z = \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} P_m \right| < \infty .$$

Now, by writing $b_m = 1/((m+1)(\log(m+2))^2)$, we have

$$\begin{aligned} Z \leqslant & |\varphi(p)| \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin mp}{m+1} \right| \\ & + \int_0^p \frac{|R(t)|}{t} \mathrm{d}t \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin mt}{m+1} \right| \\ & + \int_0^p \left| \mathrm{d}\left(R(t) \log \frac{1}{t} \right) \right| \left(\log \frac{1}{t} \right)^{-1} \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin mt}{m+1} \right| \\ & + O\left\{ |R(p)| \log \frac{1}{p} \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} b_m \right| \right\} \\ & + O\left\{ \int_0^p \left| \mathrm{d}\left(R(t) \log \frac{1}{t} \right) \right| \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} b_m \right| \right\} \\ \leqslant & Z_1 + Z_2 + Z_3 + Z_4 + Z_5 , \quad \text{say} . \end{aligned}$$

Now, for some fixed p in $0 < p < 1$, $\varphi(p) \log 1/p$ is a finite and by the hypotheses

$$R(p) \log \frac{1}{p}, \quad \int_0^p \frac{|R(t)|}{t} \log \frac{1}{t} \mathrm{d}t \quad \text{and} \quad \int_0^p \left| \mathrm{d}\left(R(t) \log \frac{1}{t} \right) \right|$$

are finite, therefore the boundedness of Z_i $(i = 1, 2, 3)$ follow from Lemma 3. And the boundedness of Z_i $(i = 4, 5)$ follow from Lemma 1 since $\sum b_m \in |C, 0|$ implies $\sum b_m \in |E, q|$ $(0 < q < 1)$ which further implies

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} b_m \right| < \infty .$$

This completes the proof of Theorem 1.

4.2. – Proof of Theorem 2. Proceeding as in Theorem 1, we only require to prove $\sum P_n \in |E, q|$ $(0<q<1)$, where, by (4.1.1),

$$P_n = \varphi_1(p)\frac{\sin np}{n} - \int_0^p \frac{\sin nt}{nt} R(t)\,\mathrm{d}t + \int_0^p R(t)\cos nt\,\mathrm{d}t$$

$$= \varphi(t)\frac{\sin np}{n} - \int_0^p \left(\mathrm{d}R(t) + \frac{R(t)}{t}\,\mathrm{d}t\right)\frac{\sin nt}{n},$$

integration by parts and using (1.7).

The series $\sum P_n \in |E, q|$ $(0<q<1)$, if, by (1.1) and (1.2),

$$Z = \sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m} P_m\right| < \infty .$$

Now,

$$Z \leqslant |\varphi(p)| \sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}\frac{\sin mp}{m+1}\right|$$

$$+ \int_0^p \left(\frac{|R(t)|}{t}\,\mathrm{d}t + |\mathrm{d}R(t)|\right)\sum (q+1)^{-n}\cdot\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}\frac{\sin mt}{m+1}\right| =$$

$$= Z_1 + Z_2, \quad \text{say}.$$

Since, by Lemma 3

$$\sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}\frac{\sin mp}{m+1}\right| = O\left(\log\frac{1}{p}\right) = O(1),$$

and $|\varphi(p)|$ is finite, therefore the boundedness of Z_1 follows. Again, by using Lemma 3, the boundedness of Z_2 follows since by the hypotheses

$$\int_0^p \frac{|R(t)|}{t}\log\frac{1}{t}\,\mathrm{d}t < \infty, \quad \int_0^p \log\frac{1}{t}\,|\mathrm{d}R(t)| < \infty .$$

This completes the proof of the theorem.

4.3. – Proof of Theorem 3. We have

$$\frac{\pi}{2} B_n(x) = \int_0^\pi \psi(t) \sin nt \, dt = \left(\int_0^p + \int_p^\pi \right) \psi(t) \sin nt \, dt = P_n + Q_n .$$

The series $\sum Q_n \in |E, q|$ $(0 < q < 1)$, by Lemma 2. Therefore, for the proof of the theorem, we only require to prove $\sum P_n \in |E, q|$ $(0 < q < 1)$. Now

$$P_n = \int_0^p \psi(t) \sin nt \, dt = \int_0^p \psi(t) \left(\sin nt + \frac{\cos nt}{nt} \right) dt - \int_0^p \psi(t) \frac{\cos nt}{nt} dt$$

$$= - \int_0^p t\psi(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt + \int_0^p \psi(t) \, dt \int_t^{\pi/2n+\pi} \frac{\partial}{\partial u} \left(\frac{\cos nu}{nu} \right) du$$

$$= - \int_0^p t\, \psi(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt - \psi_1(p) \frac{\cos np}{n} + \int_0^p \psi(t) \, dt \int_t^p \frac{\partial}{\partial u} \left(\frac{\cos nu}{nu} \right) du$$

$$= - \int_0^p t\psi(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt - \psi_1(p) \frac{\cos np}{n} + \int_0^p t\psi_1(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt ;$$

changing the order of integration and using (1.6) and (1.8),

$$P_n = \int_0^p S(t) \left(\sin nt + \frac{\cos nt}{nt} \right) dt - \psi_1(p) \frac{\cos np}{n}$$

$$= \int_0^p \frac{\cos nt}{nt} S(t) \, dt - \psi_1(p) \frac{\cos np}{n} + \int_0^p S(t) \sin nt \, dt$$

$$= \int_0^p \frac{\cos nt}{nt} S(t) \, dt - \psi(p) \frac{\cos np}{n} + \int_0^p \frac{\cos nt}{n} \, dS(t)$$

$$= \int_0^p \left(\frac{S(t)}{t} dt + dS(t) \right) \frac{\cos nt}{n} - \psi(p) \frac{\cos np}{n} .$$

The series $\sum P_n \in |E, q|$ $(0 < q < 1)$, if, by (1.1) and (1.2),

$$X = \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} P_m \right| < \infty .$$

Now

$$X \leqslant |\psi(p)| \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\cos mp}{m+1} \right| +$$

$$+ \int_0^p \left(\frac{|S(t)|}{t} \mathrm{d}t + |\mathrm{d}S(t)| \right) \sum (q+1)^{-n} \cdot \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\cos mt}{m+1} \right|$$

$$\leqslant O\left\{ |\psi(p)| \log \frac{1}{p} \right\} + O\left\{ \int_0^p \frac{|S(t)|}{t} \log \frac{1}{t} \, \mathrm{d}t \right\} + O\left\{ \int_0^p \log \frac{1}{t} \, |\mathrm{d}S(t)| \right\} = O(1)$$

by the hypotheses and due to the fact that for some fixed p in $0 < p < 1$, $\psi(p) \log 1/p$ is finite.

This completes the proof of Theorem 3.

References

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[3] R. Mohanty and S. Mohapatra, *On the $|E, q|$ summation of Fourier series and its allied series*, J. Indian Math. Soc. **32** (1968), 131-140.

Summary

The author has investigated some new criteria for absolute $|E, q|$ Summability of the Fourier series and its conjugate series at a point.

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