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Asymptotic stability for the non-linear abstract Cauchy problem (**)

1. - Introduction.

We shall consider the non-linear Cauchy problem

$$(1) \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t, u) , \qquad u(t_0) = u_0 \in D(A(t_0)) ,$$

where $f \in C(R^+ \times Y, Y)$ and Y is a Banach space. The proofs of many results in the theory of stability and boundedness rests on dividing the neighbourhood of some types of invariant sets into suitable subsets and then proving that solutions cannot leave such sets (in the case of stability) or to estimate the escape time (in the case of asymptotic stability). In [1] we obtained some global results which give a set of sufficient conditions for preventing the solutions of (1) which start in a given subset of Y from passing through any given part of its boundary. The main results were also employed to deal with various problems of stability and boundedness criteria for the abstract Cauchy problem (1). The main results are however inadequate for obtaining asymptotic stability results.

In this paper, by means of several Lyapunov functions and the theory of differential inequalities we shall discuss some global results of general character, which give a set of sufficient conditions for the solutions of (1) which start in a given subset of Y, to reach another given subset of Y in a

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finite time and remain there for all future time. We then give as applications, some results on the asymptotic stability of solutions of the abstract Cauchy problem (1).

2. - Main results.

We assume that for each $t \in R^+$; $A(t)$ is a linear operator on Y with $D(A(t))$ depending on t. As in [**1**] a solution of (1) is a strongly differentiable function $u(t) \in D(A(t))$ which satisfies (1) for all $t \geqslant t_0$. We shall assume in this paper the existence of solutions $u(t, t_0, u_0)$ of the Cauchy problem (1) for all $t \geqslant t_0$.

The theorem that follow offers a set of general conditions for the type of behaviour described in the introduction.

Theorem 2.1. *Assume that*

(i) $V\colon R^+ \times Y_0 \to R^n$ *is continuous and, for* $(t, u_1), (t, u_2) \in R^+ \times Y_0$,

$$|V(t, u_1) - V(t, u_2)| \leqslant L(t)\|u_1 - u_2\|,$$

where $L(t) \geqslant 0$ *and continuous on* R^+ *and* $Y_0 \subset Y$ *is open;*

(ii) $g \in C(R^+ \times R^n, R^n)$, $g(t, r)$ *is quasi-monotone non-decreasing in* r *for each* $t \in R^+$ *and, for* $(t, u) \in R^+ \times Y_0$, $D^+ V(t, u) \leqslant g(t, V(t, u))$;

(iii) *there exist sets* A, B *such that* $A \subset Y$, $B \subset R^n$ *and* $u \in A$ *implies* $w = V(t, u) \in B$;

(iv) *there exists a set* $F \subset Y_0$ *such that* $F \cap A \neq \phi$ *and* $u_0 \in F \cap A$ *implies* $u(t, t_0, u_0) \in Y_0$ *for* $t \geqslant t_0$;

(v) *the set* $H \subset Y_0$ *is such that* $\bar{H} \subset Y_0$ *and, for* $(t, u) \in R^+ \times (Y_0 \sim H)$, $\sum_{i=1}^{n} V_i(t, u) \geqslant \alpha(t)$, *where* $\alpha \in C[R^+, R]$;

(vi) *for each* $t \in R^+$, *and all* $h > 0$ *(*h *sufficiently small) the operator* $R[h; A(t)] = [I - hA(t)]^{-1}$ *exists as a bounded operator defined on* Y *and, for each* $u \in Y$,

$$\lim_{h \to 0} R[h; A(t)]u = u\ ;$$

(vii) *for each* $(t_0, r_0) \in R^+ \times R^n$, *the maximum solution*, $r(t, t_0, r_0)$ *of the auxiliary differential system*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t, y), \qquad y(t_0) = y_0 \tag{2}$$

exists for $t \geq t_0$ *and for any solution* $y(t, t_0, y_0)$ *of* (2) *there exists* $\tau = \tau(t_0, y_0) > 0$ *such that, if* $y_0 \in B \subset R^n$,

$$\sum_{i=1}^{n} y_i(t, t_0, y_0) < \alpha(t) \qquad (t \geq t_0 + \tau).$$

Then there exists a $T = T(t_0, u_0) > 0$ *such that* $u(t, t_0, u_0) \in H$ *for* $t \geq t_0 + T$ *whenever* $u_0 \in F \cap A$.

Proof. Let $u_0 \in F \cap A$ so that by (iv) $u(t, t_0, u_0) \in Y_0$, for all $t \geq t_0$. Moreover, setting $y_0 = V(t_0, u_0)$ we have by (iii), $y_0 \in B$. Define $m(t) = V(t, u(t, t_0, u_0))$, then $m(t_0) = y_0$ and for sufficiently small $h > 0$

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, u(t+h)) - V(t, u(t)) \\ &= V(t+h, u(t+h)) - V(t+h, R[h; A(t)]u(t) + hf(t, u)) + \\ &\quad + V(t+h, R[h; A(t)]u(t) + hf(t, u)) - V(t, u). \end{aligned}$$

By (i)

$$\begin{aligned} \frac{m(t+h) - m(t)}{h} &\leq \frac{L(t+h)}{h} \|u(t+h) - (R[h; A(t)]u(t) + hf(t, u))\| + \\ &\quad + \frac{1}{h}[V(t+h, R[h; A(t)]u + hf(t, u)) - V(t, u)]. \end{aligned}$$

Using (vi), and, for each $u \in D(A(t))$, $R[h; A(t)][I - hA(t)]u = u$, and

$$R[h; A(t)]u + hf(t, u) = u + h[A(t)u + f(t, u)] + h[R[h; A(t)]A(t)u - A(t)u].$$

Proceding in the same way as in the proof of theorem 2.2 of [**1**], and using condition (ii) we have

$$D^+ m(t) \leq g(t, m(t)). \tag{3}$$

Hence hypothesis (i), (vii) and inequality (3) imply by corollary 1.7.1 of [**2**], that

$$m(t) \leq r(t, t_0, y_0) \tag{4}$$

as far as $u(t, t_0, u_0)$ exists to the right of t_0, where $r(t, t_0, y_0)$ is the maximum solution of equation (2).

Define $T(t_0, u_0) = \tau(t_0, V(t_0, u_0))$ and let $\{t_k\}$ be a sequence such that $t_k \geqslant t_0 + T$ and $t_k \to \infty$ as $k \to \infty$. Suppose that $u(t_k, t_0, u_0) \in Y_0 \sim H$, then by (v)

$$\sum_{i=1}^{n} V_i(t_k, t_0, u_0) \geqslant \alpha(t_k) . \tag{5}$$

Moreover by (4) and (vii)

$$\sum_{i=1}^{n} V_i(t_k, u(t_k, t_0, u_0)) \leqslant \sum_{i=1}^{n} r_i(t_k, t_0, y_0) < \alpha(t_k) ,$$

which is a contradiction, hence the theorem is proved.

The following is a special case of Theorem 2.1 which is sufficient for many applications in obtaining asymptotic stability theorems of the Cauchy problem (1).

Theorem 2.2. *Assume that*

(i) $V \in C(R^+ \times Y_0, R)$ *and* $V(t, u)$ *is locally Lipschitzian in* u;

(ii) $g \in C(R^+ \times R, R)$ *and, for* $(t, u) \in R^+ \times Y_0$, $D^+ V(t, u) \leqslant g(t, V(t, u))$;

(iii) *there exists a set* $F \subset Y_0$ *such that* $u_0 \in F$ *implies* $u(t, t_0, u_0) \in Y_0$, $(t \geqslant t_0)$;

(iv) *the set* $H \subset Y_0$ *is such that* $\bar{H} \subset Y_0$ *and, for* $(t, u) \in R^+ \times (Y_0 \sim H)$, $V(t, u) \geqslant \alpha(t)$, *where* $\alpha \in C(R^+, R)$;

(v) *for each* $t \in R^+$, *all* $h > 0$, *the operator* $R[h; A(t)] = [I - hA(t)]^{-1}$ *exists as a bounded operator defined on* Y *and, for each* $u \in Y$

$$\lim_{h \to 0} R[h; A(t)]u = u ;$$

(vi) *there exists a* $\tau = \tau(t_0, u_0) > 0$ *such that for any solution* $y(t, t_0, y_0)$ *of the scalar differential equation*

$$\frac{dy}{dt} = g(t, y) , \qquad y(t_0) = y_0 , \tag{6}$$

the relation $y(t, t_0, y_0) < \alpha(t)$, $t \geqslant t_0 + \tau$, *holds. Then there exists a* $T = T(t_0, u_0) > 0$ *such that* $u_0 \in F$ *implies* $u(t, t_0, u_0) \in H$ *for* $t \geqslant t_0 + T$.

3. - Applications of main results.

We give in this section some applications of our results to asymptotic stability criteria for the abstract Cauchy problem (1).

A subset $M \subset Y$ is said to be *self-invariant with respect to the system* (1) if $u_0 \in M$ implies $u(t) \in M$ for $t \geqslant t_0 \geqslant 0$. Suppose M is a compact subset of Y and suppose that it is self-invariant with respect to the abstract Cauchy problem (1) and for $\varrho > 0$, define

$$S(M, \varrho) = \{u \in Y;\ d(u, M) < \varrho\}\,,$$

where $d(u, M) = \inf_{y \in M} \|u - y\|_Y$.

We now state a theorem which gives sufficient conditions for the asymptotic stability of a self-invariant set M with respect to the Cauchy problem (1).

Theorem 3.1. *Assume that*

(i) $V \in C(R^+ \times S(M, \varrho) \sim M, R)$, *$V(t, u)$ is locally Lipschitzian in u and $V(t, u) \to -\infty$, when $d(u, M) \to 0$ for each $t \in R^+$,*

(ii) $b \in C(R^+ \times (0, \varrho), R)$ *and for* $(t, u) \in R^+ \times S(M, \varrho) \sim M$, $V(t, u) \geqslant b(t, d(u, M))$;

(iii) $g \in C(R_+ \times R, R)$ *and for* $(t, u) \in R^+ \times S(M, \varrho) \sim M$, $D^+ V(t, u) \leqslant g(t, V(t, u))$;

(iv) *for each $t \in R^+$ and all $h > 0$ (h small) the operator $R[h; A(t)]$ exists as a bounded operator defined on Y and, for each $u \in Y$,*

$$\lim_{h \to 0^+} R[h; A(t)]u = u$$

(v) *every solution $y(t, t_0, y_0)$ of the scalar differential equation* (6) *satisfies $y(t, t_0, y_0) < b(t, r)$, $t \geqslant t_0$, for every $r \in (0, \varrho)$ provided $u_0 < b(t_0, r)$;*

(vi) *$b(t, w)$ is non decreasing in w for each $t \in R^+$ and there exists a $\tau = \tau(t_0, y_0) > 0$ such that every solution $y(t, t_0, y_0)$ of* (6) *satisfies the relation $y(t, t_0, y_0) < b(t, r)$, $t \geqslant t_0 + \tau$ for all $r \in (0, \varrho)$.*

Then the self-invariant set M is asymptotically stable.

Proof. By theorem 3.5 of [1], the set M is stable, hence for $\varepsilon = \varrho$, there exists $\delta_0 = \delta(t_0, \varrho)$ such that $u_0 \in S(M, \delta_0)$ implies $u(t, t_0, u_0) \in S(M, \varrho)$, $t \geqslant t_0$. Now set $F = S(M, \delta_0)$, then the condition (iii) of Theorem 2.2 is satisfied.

Let $t_0 \in R^+$ and $0 < \varepsilon < \varrho$ and set $H = S(M, \varepsilon)$, $Y_0 = S(M, \varrho) \sim M$, then for $(t, u) \in Y_0 \sim H$ and the monotonicity of $b(t, r)$, $V(t, u) \geqslant b(t, \varepsilon)$. Choosing $\alpha(t) = b(t, \varepsilon)$, then condition (iv) of Theorem 2.2 is satisfied. Conditions (i), (ii), (v) and (vi) of Theorem 2.2 are already part of the hypothesis, hence by the conclusions of that theorem, there exists a $T = T(t_0, u_0) > 0$ such that $u_0 \in F$ implies $u(t, t_0, u_0) \in H$ for $t \geqslant t_0 + T$, which is the asymptotic stability of M.

The following theorem gives sufficient conditions for the conditional asymptotic stability of the set M with respect to (1). For the definition of conditional stability of a sel-invariant set M with respect to the abstract Cauchy problem (1), see [1].

Theorem 3.2. *Assume that*

(i) $V \in C(R^+ \times S(M, \varrho), R^n)$ *and* $V(t, u)$ *is locally Lipschitzian in* u;

(ii) $g \in C(R^+ \times R^n, R^n)$, $g(t, x)$ *is quasi-monotone non-decreasing in* x *for each* $t \in R^+$ *and for* $(t, u) \in R^+ \times S(M, \varrho) \sim M$, $D^+ V(t, u) \leqslant g(t, V(t, u))$;

(iii) *there exist sets* E_1, E_2 *such that* $M \subset E_1 \subset Y$, $\{0\} \subset E_2 \subset R^n$ *and* $u \in E_1$ *implies* $w = V(t, u) \in E_2$;

(iv) $b \in C(R \times (0, \varrho), R)$ *and for* $(t, u) \in R^+ \times S(M, \varrho) \sim M$, $\sum_{i=1}^{n} V_i(t, u) \geqslant b(t, d(u, M))$;

(v) *whenever* $u \in M$, *for* $(t, r) \in R^+ \times (0, \varrho)$, $\sum_{i=1}^{n} V_i(t, u) < b(t, r)$;

(vi) *every solution* $y(t, t_0, y_0)$ *of the differential equation* (2) *satisfies* $\sum_{i=1}^{n} y_i(t, t_0, y_0) < b(t, r)$, $t \geqslant t_0$, *for all* $r \in (0, \varrho)$, *provided* $y_0 \in E_2$ *and* $\sum_{i=1}^{n} y_{i_0} < b(t_0, r)$;

(vii) *for each* $t \in R^+$ *and* $h > 0$ $R[h; A(t)]$ *exists as a bounded operator defined on* Y *and for* $u \in Y$

$$\lim_{h \to 0^+} R[h; A(t)]u = u ;$$

(viii) $b(t, y)$ *is non-decreasing in* y *for each* $t \in R^+$ *and there exists a*

$\tau = \tau(t_0, y_0) > 0$ *such that for any solution* $y(t, t_0, y_0)$ *of* (2) *the relation* $\sum_{i=1}^{n} y_i(t, t_0, y_0) < b(t, r)$, $t \geqslant t_0 + \tau$, *holds for every* $r \in (0, \varrho)$. *Then the self-invariant set* M *is conditionally asymptotically stable.*

Proof. We check that all conditions of Theorem 2.1 are satisfied. By theorem 3.3 of [1] the set M is conditionally stable, hence for $\varepsilon = \varrho$, $\exists \delta_0 = \delta(t_0, \varrho)$ and a subset $N \subset M$ such that $u_0 \in S(N, \delta_0)$ implies $u(t, t_0, u_0) \in S(M, \varrho)$, $t \geqslant t_0$. Now set $F = S(N, \delta_0)$ then again condition (iii) of Theorem 2.2 is satisfied.

Let $t_0 \in R^+$ and $0 < \varepsilon < \varrho$. Set $H = S(M, \varepsilon)$, $Y_0 = S(M, \varrho) \sim M$, then by (viii) and (iv), for $(t, u) \in Y_0 \sim H$, $\sum_{i=1}^{n} V_i(t, u) \geqslant b(t, \varepsilon)$. Setting $\alpha(t) = b(t, \varepsilon)$ then all conditions of Theorem 2.1 are now satisfied and so the conclusion of that theorem implies the conditional asymptotic stability of the self-invariant set M.

We state a theorem of asymptotic stability of a conditionally-invariant set whose proof can be reduced to Theorem 2.2. Let M and N be two subsets of Y such that $M \subset N$. For the definition of a conditionally invariant set N on M and its stability see [1].

Theorem 3.3. *Assume that*

(i) $V \in C(R^+ \times S(N, \varrho), R)$ *and* $V(t, u)$ *is locally Lipschitzian in* u;

(ii) $g \in C(R^+ \times R, R)$ *and for* $(t, u) \in R^+ \times S(N, \varrho) \sim M$, $D^+ V(t, u) \leqslant g(t, V(t, u))$;

(iii) $b \in C(R^+ \times (0, \varrho), R)$, $b(t, x)$ *is non-decreasing in* x, *for each* $t \in R^+$ *and, for* $(t, u) \in R^+ \times S(N, \varrho) \sim M$, $V(t, u) \geqslant b(t, d(u, N))$;

(iv) *there exists a* $\tau = \tau(t_0, y_0) > 0$ *such that any solution* $y(t, t_0, y_0)$ *of* (6) *satisfies the inequality* $y(t, t_0, y_0) < b(t, r)$, $t \geqslant t_0 + \tau$ *for every* $r \in (0, \varrho)$;

(v) *for each* $t \in R^+$ *and* $h > 0$, $R[h; A(t)]$ *exists as a bounded operator defined on* Y *and for* $u \in Y$

$$\lim_{h \to 0} R[h; A(t)]u = u .$$

Then the stability of the conditionally invariant set N *on* M *implies the asymptotic stability of* N *on* M.

References

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Sunto

Si ottengono alcuni risultati su l'asintotica stabilità delle soluzioni del problema astratto di Cauchy.

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