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Another real inversion theorem for a distributional ${}_1F_1$-transform (**)

1. - Introduction.

A generalization of the conventional Laplace transform

$$(1.1)\qquad F(x)=\int_0^\infty e^{-xy}\, f(y)\,\mathrm{d}y$$

had been given by Erdelyi [2] in the form

$$(1.2)\quad F(x)=\frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\int_0^\infty (xy)^{\beta}\,{}_1F_1(\beta+\eta+1;\,\alpha+\beta+\eta+1;\,-xy)\,f(y)\,\mathrm{d}y.$$

For $\alpha=\beta=0$ (1.2) reduces (1.1).

Two real inversion formulas for this transform were given by Joshi $[\mathbf{3}]_{1,2,3}$. Another real inversion formula was also given by him ($[\mathbf{3}]_1$, chapter V). The object of this paper is to extend this inversion formula for the two-sided case to distributions and finally deduce the uniqueness theorem also for the same transform.

The generalized Laplace transform (1.2) of a distribution $f(y)$ in $0<y<\infty$ can be defined in the distributional sense, as an application of $f(y)$ to

$$\frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\,(xy)^{\beta}\,{}_1F_1(\beta+\eta+1;\,\alpha+\beta+\eta+1;\,-xy)$$

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in the following way

$$(1.3)\quad F(x) = \langle f(y), \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}(xy)^{\beta}\,{}_1F_1(\beta+\eta+1;\alpha+\beta+\eta+1;-xy)\rangle\ .$$

We define $F(x)$ as ${}_1F_1$*-transform of* $f(y)$ and the transformation is called ${}_1F_1$*-transformation.* A number of inversion formulas in the conventional Laplace transform theory [**1**], [**7**], has recently been extended to Schwartz's distributions by Zemanian [**8**]$_2$.

The inversion formula proved in ([**3**]$_1$, chapter V) is the following

Theorem. *If* $f(y) \in L$ *in* $0 \leqslant y \leqslant R$ *for every positive* R *if* $f(y) = 0$ *in* $-\infty < y < 0$ *and is such that the integral* (1.2) *converges for some* x $(=x_0)$ *say, then* $\lim_{n\to\infty} Q_{n,t}[F(x)] = f(t)$ *for all positive* t *in the Lebesgue set for* $f(y)$ *where*

$$(1.4)\qquad Q_{n,t}[F(x)] = \frac{(-1)^n}{\Gamma(n+1+\beta-\alpha)}\, x^{-\eta+1}\, \mathscr{D}^n x^{-\alpha}\, \mathscr{D}^{-n} x^{\alpha+\beta+\eta+n}\, \mathscr{D}^n[x^{-\beta}F(x)]_{x=n/t}\ .$$

2. - Notations and terminology.

A function is said to be *smooth* if all its derivatives of all orders are continuous at all points of its domain. The space of testing functions denoted by $\mathscr{D}$ consists of all complex-valued functions $\varphi(t)$ that are smooth and zero outside some finite interval.

R_t denotes the one-dimensional euclidean space consisting of all real values for t. $R_{x,y}$ is the two-dimensional euclidean space consisting of all real pairs (x, y). $\mathscr{D}_\tau$ is the space $\mathscr{D}$ of testing function which are defined over R_τ and $\mathscr{D}_{x,u}$ is the space $\mathscr{D}$ of testing functions defined over $R_{x,u}$.

Let I denote an open interval $(0, \infty)$ on the real line. $\mathscr{D}_I$ is the space of smooth functions on I having compact supports with respect to I. $\mathscr{D}_I'$ represents the dual space of $\mathscr{D}_I$. The sequence of distributions $\{f_\nu\}_{\nu=1}^{\infty}$ is said to converge in $\mathscr{D}_I'$ if, for every $\varphi \in \mathscr{D}_I$, the sequence of numbers $\{\langle f_\nu, \varphi\rangle\}_{\nu=1}^{\infty}$ converges in the ordinary sense of convergence of numbers.

Let A and B be two real numbers $(A \leqslant 0 \leqslant B)$ and t be a real variable. We define $K_{A,B}(t)$ to be the function given by

$$K_{A,B}(t) = e^{At} \qquad (0 \leqslant t < \infty)\,, \qquad\qquad K_{A,B}(t) = e^{Bt} \qquad (-\infty < t < 0)\,.$$

$L_{A,B}$ is defined as the space of all complex-valued smooth functions $\varphi(t)$ on $-\infty < t < \infty$ on which the functionals γ_k defined by

$$\gamma_k(\varphi) = \sup_{-\infty<t<\infty} |K_{A,B}(t) D_t^k \varphi(t)| \qquad (k = 0, 1, 2, \ldots)$$

assume finite values. We assign to $L_{A,B}$ the topology generated by $\{\gamma_k\}_{k=0}^{\infty}$ thereby making it countably multinormed space. $L_{A,B}$ is sequentially complete and hence a Frechet space. $L'_{A,B}$ denotes the space of continuous linear functionals on $L_{A,B}$. The number that $f \in L'_{A,B}$ assigns to $\varphi \in L_{A,B}$ is denoted by $\langle f, \varphi\rangle = \langle f(t), \varphi(t)\rangle$.

Throughout this paper it will be assumed that $\operatorname{Re}\beta \geqslant 0$, $\operatorname{Re}\eta > 0$, $\operatorname{Re}(\alpha + \beta + \eta + 1) \neq 0, -1, -2, \ldots$; $a = \beta + \eta + 1$, $b = \alpha + \beta + \eta + 1$ and $P = \Gamma(a)/\Gamma(b)$. We observe that $\mathscr{D} \subset L_{A,B}$ ([**8**]$_3$, p. 51, [**4**]) and $L'_{A,B} \subset \mathscr{D}'$ ([**8**]$_2$, p. 160, [**4**]).

A distribution f is said to be ${}_1F_1$*-transformable* if there exist two real numbers A, B $(A \leqslant 0 \leqslant B)$ such that $f \in L'_{A,B}$. σ_1 is defined as the infimum of all A for which $f \in L'_{A,B}$.

3. – Theorem 3.1. *Let f be a ${}_1F_1$-transformable distribution whose support is contained in I. Let the integro-differential operator* (1.4) *be written as*

$$P_{n,t}(D)F = \frac{(-1)^n}{\Gamma(\beta + n + 1 - \alpha)}\, x^{-\eta+1}\, \mathscr{D}^n x^{-\alpha}\, \mathscr{D}^{-n} x^{\alpha+\beta+\eta+n}\, \mathscr{D}^n\, [x^{-\beta} F(x)]_{x=n/t}\,,$$

where $F(x)$ is given by (1.3). *Then, in the sense of convergence in $\mathscr{D}'_I$*

$$f(t) = \lim_{n\to\infty} P_{n,t}(D)\, [F(x)]_{x=n/t}\,,$$

that is, for every $\varphi \in \mathscr{D}_I$,

$$\lim_{n\to\infty} \langle P_{n,t}(D)\, [F(x)]_{x=n/t}\,, \varphi(t)\rangle = \langle f(t), \varphi(t)\rangle\,,$$

provided $\operatorname{Re}\beta + n + 1 > \operatorname{Re}\alpha$.

Proof. Let the support of $\varphi(t)$ be contained in $c_1 \leqslant t \leqslant c_2$ where $0 < c_1 < c_2 < \infty$. Here $F(x)$ is a smooth function ([**4**], theorem 4.1). This fact was proved there by using Cauchy's integral formula. Hence $P_{n,t}(D)F(x)$ is also a smooth function so that $\langle P_{n,t}(D)F, \varphi\rangle$ is an integral. For any $\varphi \in \mathscr{D}_I$, by repeated integrations by parts and a change of variable ($t = n/x$) we have

$$(3.1)\qquad \langle P_{n,t}(D)[F(x)], \varphi(t)\rangle = \langle P_{n,x}(D)[F(x)], \varphi(n/x)\rangle$$

$$(3.2)\qquad = \langle F(x), P_{n,x}(-D)\varphi(n/x)(-n/x^2)\rangle =$$

$$(3.3)\qquad = \langle P_{n,x}(-D)\varphi(n/x)\,(-n/x^2),\ \langle f(y), P(xy)^{\beta}\,{}_1F_1(a; b; -xy)\rangle\rangle\ .$$

We will now justify the above steps. (3.1) equals (3.2) by the usual definitions for the shifting of operator and differentiation of distributions. Since $\varphi \in \mathscr{D}_I$, $-(n/x^2)P_{n,x}(-D)\varphi(n/x)$ also lies in $\mathscr{D}_I$ with its support in $n/c_2 \leqslant x \leqslant n/c_1$ and $F(x)$ is a smooth function in $\sigma_1 < x < \infty$. Hence (3.2) equals (3.3). Since $f(y) \in L'_{A,B}$ and $L'_{A,B} \subset \mathscr{D}'$, $f(y) \in \mathscr{D}'_y$ and the testing function in (1.3) belongs to $\mathscr{D}_{x,y}$. Hence $-(n/x^2)P_{n,x}(-D)\varphi(n/x)$ in (3.3) belongs to $\mathscr{D}'$. So ([**8**]$_1$, Theorem 5.3-2) we can change the order of inner product in (3.3) and write (3.3) as

$$\langle f(y), \langle P_{n,x}(-D)\varphi\left(\frac{n}{x}\right)\left(-\frac{n}{x^2}\right), P(xy)^{\beta}\,{}_1F_1(a; b; -xy)\rangle\rangle =$$

$$= \langle f(y), \langle -\frac{n}{x^2}\varphi\left(\frac{n}{x}\right),\ P_{n,x}(D)\ \{P(xy)^{\beta}\,{}_1F_1(a;\ \ b; -xy)\}\rangle\rangle$$

$$(3.4)\qquad = \langle f(y),\ \varrho(y, n)\rangle,$$

where

$$\varrho(y, n) = \langle -\frac{n}{x^2}\ \varphi(n/x),\ P_{n,x}(D)\ \{P(xy)^{\beta}\,{}_1F_1(a; b; -xy)\}\rangle\ .$$

(3.4) has a sense since $f(y) \in L'_{A,B}$ and $\varrho(y, n) \in L_{A,B}$ imply that $f(y) \in \mathscr{D}'_y$ and $\varrho(y, n) \in \mathscr{D}_{x,y}$ respectively.

To complete the proof, we have to prove that as $n \to \infty$, $\varrho(y, n)$ converges to $\varphi(y)$ in $L_{A,B}$ for every A and B ($A \leqslant 0 \leqslant B$).

$$\varrho(y, n) = \frac{1}{\Gamma(n+1+\beta-\alpha)}\int_0^{\infty}\varphi(n/x)(-n/x^2)x^{-\eta+1}\,\mathscr{D}^n x^{-\alpha}\,\mathscr{D}^{-n}x^{\alpha+\beta+\eta+n}\,\mathscr{D}^n[x^{-\beta}F(x)]\,dx,$$

which by ([**3**], p. 73) is equal to

$$(3.5)\quad \frac{\Gamma(a+2n)(-n)}{\Gamma(b+2n)\Gamma(n+1+\beta-\alpha)}\int_0 \varphi(n/x)x^{\beta+n-1}y^{\beta+n}\,{}_1F_1(a+2n;\ b+2n; -xy)\,dx.$$

We can write (3.5), after changing the variables $x = u$, $n = k$; $y = x$,

$$\varrho(x, k) = \frac{\Gamma(a+2k)(-k)}{\Gamma(b+2k)\Gamma(k+1+\beta-\alpha)}\int_0^\infty \varphi(k/u)\,u^{\beta+k-1}x^{\beta+k}\,{}_1F_1(a+2k;\, b+2k;\, -ux)\,\mathrm{d}u\,.$$

Now putting $u = ky/x$, $\varrho(x, k)$ becomes

$$-\frac{\Gamma(a+2k)\,k^{\beta+k+1}}{\Gamma(b+2k)\Gamma(k+1+\beta-\alpha)}\int_0^\infty \varphi(x/y)\,y^{\beta+k-1}\,{}_1F_1(a+2k;\, b+2k;\, -ky)\,\mathrm{d}y\,.$$

We are to prove that $\varrho(x, k) \to \varphi(x)$ in $L_{A,B}$ as $k \to \infty$. In other words we have to prove that, for each nonnegative integer ν, $K_{A,B}(x)D_x^\nu[\varrho(x, k) - \varphi(x)]$ converges uniformly to zero on $0 < x < \infty$ as $k \to \infty$. By ([**5**], p. 48) we can write

$$K_{A,B}(x)D_x^\nu[\varrho(x, k) - \varphi(x)] =$$

$$= \frac{\Gamma(a+2k)\,k^{\beta+k+1}K_{A,B}(x)}{\Gamma(b+2k)\Gamma(\beta+k+1-\alpha)}\int_0^\infty y^{\beta+k}\,{}_1F_1(a+2k;\, b+2k;\, -ky)\cdot$$

$$\cdot\,\varphi^{(\nu)}(x/y)\,y^{-(\nu+1)}\,\mathrm{d}y - \frac{\Gamma(a+2k)}{\Gamma(b+2k)}k^{\beta+k+1}K_{A,B}(x)\cdot$$

$$\cdot\int_0^\infty y^{\beta+k}\,{}_1F_1(a+2k;\, b+2k;\, -ky)\,\frac{\Gamma(\alpha+\eta+k)}{\Gamma(\beta+k+1)\Gamma(\eta+k)}\,\varphi^{(\nu)}(x)\,\mathrm{d}y = I_1 + I_2\,.$$

But since $K_{A,B}(x)\varphi^{(\nu)}(x) < C_1$

$$|I_2| \leqslant \frac{\Gamma(a+2k)\Gamma(\alpha+\eta+k)\,C_1k^{\beta+k+1}}{\Gamma(b+2k)\Gamma(\beta+k+1)\Gamma(\eta+k)}\left[\int_0^{1-\delta} y^{\beta+k}\,{}_1F_1(a+2k;\, b+2k;\, -ky)\,\mathrm{d}y + \right.$$

$$\left. + \int_{1-\delta}^{1+\delta} \int_{1+\delta}^{\infty}\right] = I_2' + I_2'' + I_2''' \quad (0 < \delta < 1)\,,$$

$$I_2' = C_1\frac{\Gamma(a+2k)\Gamma(\alpha+\eta+k)\,k^{\beta+k+1}}{\Gamma(b+2k)\Gamma(\beta+k+1)\Gamma(\eta+k)}\int_0^{1-\delta}\frac{(ky)^{b+2k-1}}{k^{b+2k}}\,\frac{1}{y^{\alpha+\eta+k}}\,e^{-ky}\,{}_1F_1(\alpha;\, b+2k;\, ky)\,k\,\mathrm{d}y\,,$$

by using Kummer's formula ([**5**], p. 6).

Putting $ky=t$ and noting that $\operatorname{Sup}\left(\frac{1}{y^{\alpha+\eta+k}}\right)<M$ for $0<y<1-\delta$ we have

$$I_2' \leqslant MC_1 \frac{\Gamma(a+2k)\Gamma(\alpha+\eta+k)}{\Gamma(b+2k)\Gamma(\beta+k+1)\Gamma(\eta+k)} k^{\beta+1-b-k} \int_0^{k(1-\delta)} t^{b+2k-1} e^{-t} {}_1F_1(\alpha;\, b+2k;\, t)\, dt$$

$$\leqslant \frac{MC_1\Gamma(a+2k)\Gamma(\alpha+\eta+k)}{\Gamma(b+2k)\Gamma(\beta+k+1)\Gamma(\eta+k)} k^{\beta+1-b} \left[\frac{\{k(1-\delta)\}^{b+2k}}{(b+2k)e^{k(1-\delta)}} \times \right.$$

$$\left. \times\, {}_1F_1(\alpha+1;\, b+2k+1;\; k(1-\delta)\right],$$

by using a result ([5], p. 42 (3.2.5)). Again by using the results ([5], (4.3.7), p. 66; [6], p. 253) the right hand side of the above inequality is asymptotic to

$$\frac{MC_1 k^k}{(b+2k)} \left[\frac{(1-\delta)^{b+2k}(\delta)^{-(\alpha+1)}}{e^{k(1-\delta)}} \left\{1 - \frac{(\alpha+1)(\alpha+2)}{2k}\left(\frac{1-\delta}{\delta}\right)^2\right\}\right].$$

This proves that $I_2' \to 0$ as $k\to\infty$. Similarly I_2'' can also be shown to tend to 0 as $k\to\infty$. Coming back to I_1

$$I_1 = \int_0^{1-\delta} + \int_{1-\delta}^{1+\delta} + \int_{1+\delta}^{\infty} = I_1' + I_1'' + I_1'''.$$

Here

$$I_1' \leqslant C_2 \frac{\Gamma(a+2k)k^{\beta+k+1}}{\Gamma(b+2k)\Gamma(\beta+k+1-\alpha)} \int_0^{1-\delta} \frac{(ky)^{b+2k-1}}{k^{b+2k-1}} \frac{y^{\beta+k}}{y^{\nu+b+2k}} e^{-ky} {}_1F_1(\alpha;\, b+2k;\, ky)\, dy,$$

since $K_{A,B}(x)\varphi^{(\nu)}(x/y) < C_2$ in $0<x<\infty$. Putting $ky=t$ and noting that

$$\operatorname{Sup}\left(\frac{1}{y^{\nu+b+k-\beta}}\right) < M_1 \qquad \text{for } 0<y<1-\delta,$$

$$I_1' \leqslant \frac{M_1C_2\Gamma(a+2k)k^{\beta+1-b-k}}{\Gamma(b+2k)\Gamma(\beta+k+1-\alpha)} \int_0^{k(1-\delta)} t^{b+2k-1}\; e^{-t}\; {}_1F_1(\alpha;\, b+2k;\, t)\, dt.$$

By the use of a result ([5], p. 42) the right hand side of the above inequality reduces to

$$(3.6)\quad \frac{M_1 C_2 \Gamma(a+2k)\, k^{\beta+1-b-k}}{\Gamma(b+2k)\Gamma(\beta+k+1-\alpha)} \left[\frac{e^{-t}\, t^{b+2k}}{(b+2k)}\, {}_1F_1(\alpha+1;\, b+2k+1;\, t)\right]_0^{k(1-\delta)} .$$

By considering the asymptotic properties of gamma function ([6], p. 253) and ${}_1F_1$ function ([5], p. 66) for large values of k, we find that (3.6) is asymptotic to

$$k^k M_1 C_2 \left[\frac{(1-\delta)^{b+2k}\,(\delta)^{-(\alpha+1)}}{e^{k(1-\delta)}\,(b+2k)}\left\{1-\frac{(\alpha+1)(\alpha+2)}{2k}\left(\frac{1-\delta}{\delta}\right)^2\right\}\right],$$

which obviously tends to zero as $k\to\infty$ in $0<y<1-\delta$. In a similar way we can prove that $I_1''\to 0$ uniformly as $k\to\infty$. It now remains to prove that $I_1'''+I_2'''\to 0$ as $k\to\infty$. We have

$$I_1''+I_2''=\frac{\Gamma(a+2k)}{\Gamma(b+2k)}\, k^{\beta+k+1}\,\frac{K_{A,B}(x)}{\Gamma(\beta+k+1-\alpha)}\, \cdot$$

$$\cdot \int_{1+\delta}^{\infty} y^{\beta+k}\cdot {}_1F_1(a+2k;\, b+2k;\, -ky)\left\{\frac{\varphi^{(\nu)}(x/y)}{y^{\nu+1}}-\frac{\Gamma(\beta+k+1-\alpha)}{\Gamma(\beta+k+1)}\, \cdot \right.$$

$$\left. \cdot\, \frac{\Gamma(\alpha+\eta+k)}{\Gamma(k+\eta)}\,\varphi^{(\nu)}(x)\right\} dy .$$

We shall prove that $I_1'''+I_2'''\to 0$ uniformly as $k\to\infty$ in any $a_1\leqslant x\leqslant b_1$, provided $\beta+k-\nu+1>0$.

Let

$$\lambda(\vartheta,\, x)=\int_{1+\delta}^{\vartheta}\left\{\varphi^{(\nu)}(x/y)\, y^{\beta+k-\nu-1}-\frac{\Gamma(\beta+k+1-\alpha)}{\Gamma(\beta+k+1)}\, y^{\beta+k}\, \cdot \right.$$

$$\left. \cdot\, \frac{\Gamma(\alpha+\eta+k)}{\Gamma(k+\eta)}\,\varphi^{(\nu)}(x)\right\} dy$$

$$= x^{\beta+k-\nu}\int_{x/1+\delta}^{x/\vartheta}\left[-\varphi^{(\nu)}(\tau)\,\tau^{(-\beta+k-\nu+1)}+x^{\nu+1}\, \cdot \right.$$

$$\left. \cdot\, \frac{\Gamma(\beta+k+1-\alpha)\Gamma(\alpha+\eta+k)}{\Gamma(\beta+k+1)\Gamma(k+\eta)}\,\tau^{-(\beta+k+2)}\,\varphi^{(\nu)}(x)\right] d\tau ,$$

after replacing x/y by τ. Since $\varphi(\tau)\in\mathscr{D}(I)$ and τ^r $(r<0)$ is continuous in

$1 + \delta < \tau < \infty$, it is clear that $\lambda(\vartheta, x)$ has an upper bound M^1 in the region $1 + \delta \leqslant \vartheta \leqslant \infty$, $a_1 \leqslant x \leqslant b_1$ with the condition $\beta + k - \nu + 1 > 0$. Consider

$$\int_{1+\delta}^{\infty} \lambda(y, x)\, \mathrm{d}_y [{}_1F_1(a + 2k;\, b + 2k;\, -ky)] =$$

$$= [\lambda(y, x)\, {}_1F_1\, (a+2k;\, b+2k;\, -ky)\,]_{1+\delta}^{\infty} - \int_{1+\delta}^{\infty} {}_1F_1\, (a + 2k;\, b + 2k;\, -ky)\, \mathrm{d}\lambda(y, x)\, \mathrm{d}y\, .$$

But by ([5], p. 60, (4.1.8))

$${}_1F_1\, (a + 2k;\, b + 2k;\, -ky) \sim \frac{\Gamma(b + 2k)}{\Gamma(\alpha)}\, (ky)^{-(a+2k)} \to 0 \qquad \text{as } y \to \infty$$

and by the definition of $\lambda(\vartheta, x)$, we have

$$I_1''' + I_2''' = -K \int_{1+\delta}^{\infty} \lambda(y, x)\, \mathrm{d}_y [{}_1F_1\, (a + 2k;\, b + 2k;\, -ky)]\, ,$$

where

$$K = \frac{\Gamma(a + 2k)\, k^{\beta+k+1} K_{A,B}(x)}{\Gamma(b + 2k)\Gamma(\beta + k + 1 - \alpha)}\, .$$

Moreover, we note that the function ${}_1F_1(a + 2k;\, b + 2k;\, -ky)$ is a decreasing function of y in $(1 + \delta, \infty)$. For

$${}_1F_1\, (a + 2k;\, b + 2k;\, -k(y + h)) - {}_1F_1\, (a + 2k;\, b + 2k;\, -ky) \qquad (h > 0)$$

is asymptotic to

$$\frac{1}{k^{a+2k}} \left[\frac{y^{a+2k} - (y + h)^{a+2k}}{\{y(y + h)\}^{a+2k}}\right] < 0\, .$$

We also note that $K_{A,B}(x)$ has an upper bound B_1 in $a_1 \leqslant x \leqslant b_1$. Hence

$$I_1''' + I_2''' \leqslant \frac{\Gamma(a + 2k)\, k^{\beta+k+1}}{\Gamma(b + 2k)}\; \frac{B_1 M^1}{\Gamma(\beta + k + 1 - \alpha)}\, [{}_1F_1\, (a + 2k;\, b + 2k;\, -ky)]_{1+\delta}^{\infty}\cdot$$

As $y \to \infty$

$$I_1''' + I_2''' \leqslant \frac{\Gamma(a + 2k)}{\Gamma(b + 2k)}\, k^{\beta+k+1}\, \frac{B_1 M^1}{\Gamma(\beta + k + 1 - \alpha)}\, \cdot \tag{3.7}$$

$$\cdot\, [{}_1F_1(a + 2k;\, b + 2k;\, -k(1 + \delta))]\, .$$

From a result ([**5**], p. 6) and asymptotic properties ([**5**], p. 66; [**6**], p. 253) the right hand side quantity of (3.7) is asymptotic to

$$\frac{k^k(-\delta)^{-\alpha}}{e^{k(1+\delta)}}\left[1-\frac{\alpha(\alpha+1)}{2k}\left(\frac{1+\delta}{-\delta}\right)^2\right],$$

which $\to 0$ as $k\to\infty$. This proves the theorem.

4. - The uniqueness theorem.

Let f and g be two ${}_1F_1$-transformable distributions defined by

$$F(x)=\langle f(y),\frac{\Gamma(a)}{\Gamma(b)}(xy)^{\beta}{}_1F_1(a;\,b;\,-xy)\rangle\,,$$

$$G(x)=\langle g(y),\frac{\Gamma(a)}{\Gamma(b)}(xy)^{\beta}{}_1F_1(a;\,b;\,-xy)\rangle\,.$$

If $F(x)=G(x)$ for all x in $0<x<\infty$, then $f=g$ in the sense of equality in $\mathscr{D}'_I$.

Proof. Let $P_{n,t}(D)[F(x)]_{x=n/t}$ be the inversion operator as specified in Theorem 3.1. Then, in the sense of convergence in $\mathscr{D}'_I$, we have from Theorem 3.1.

$$f(t)=\lim_{n\to\infty}P_{n,t}(D)[F(x)]_{x=n/t}\,,\qquad g(t)=\lim_{n\to\infty}P_{n,t}(D)[G(x)]_{x=n/t}=g,$$

so that

$$f(t)=\lim_{n\to\infty}P_{n,t}(D)[F(x)]_{x=n/t}=\lim_{n\to\infty}P_{n,t}(D)[G(x)]_{x=n/t}=g\,,$$

whence it follows that $f=g$ in the sense of equality in $\mathscr{D}'_I$.

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