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# On a class of bisimple inverse semigroups (**)

1. – A bicyclic semigroup $[\mathbf{2}]_1$ is a semigroup with identity element 1 generated by two elements $a$, $b$ such that $ab = 1 \neq ba$. It is easily seen that the elements of this semigroup are of the form $(b^m a^n)$; $m, n \geq 0$. The bicyclic semigroup belongs to the general class of bisimple inverse semigroups $[\mathbf{2}]_2$.

The definition of a bisimple inverse semigroup is rather involved and is as follows. If $x, y$ are elements of a semigroup. $S$, $x$ is said to be *left* (*right*) *equivalent to* $y$ if $SxUx = SyUy$ ($xSUx = ySUy$) and the set of all elements of $S$ that are left (right) equivalent to $x \in S$ is called *the left* (*right*) *equivalence class of* $x$ and is denoted by $L_x$ ($R_x$).

The elements $x, y \in S$ are said to be *D-equivalent* it there exists an element $z$ in $S$ such that $z \in L_x \cap R_y$ and the set of all elements of $S$ that are $D$-equivalent to $x \in S$ is called the *D-class* of $x$ and is denoted by $D_x$. If $S$ consists only of a single $D$-class, it is said to be *bisimple*. A semigroup $S$ is said to be *regular* if for each $x$ in $S$ there exists an element $y$ in $S$ such that $xyx = x$. Finally $S$ is said to *inverse* if $S$ is regular and idempotents in $S$ commute.

The notion of a bicyclic semigroup was first introduced by Lyapin [4] and has been studied by Olaf Anderson [1] and others, an account of which is given in $[\mathbf{2}]_1$ and $[\mathbf{2}]_2$. In [3] we generalized the concept of a bicyclic semigroup to a semigroup formed by $G_+ \times G_+$, where $G_+ = \{x \in G \mid x \geq 1\}$, $G$ being an ordered group, under the multiplication given by

$$(x, y)(u, v) = \begin{cases} (x, yu^{-1}v) & \text{if} \quad y \geq u\,, \\ (uy^{-1}x, v) & \text{if} \quad y \leq u\,. \end{cases}$$

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It is easily seen that for each $x \in G_+$ the set $\{(x^m, x^n) \mid m, n \geqslant 0\}$ is a bicyclic subsemigroup of $G_+ \times G_+$. Among other properties of the semigroup $G_+ \times G_+$ we showed in particular that this special class of semigroups is bisimple inverse. In this paper we give a natural representation of the above semigroup $G_+ \times G_+$.

**2.** – Let $F$ be the set of all functions $f$ on an ordered multiplicative group $G$ (not necessarily commutative) with values in an arbitrary set $S$ with more than one element such that

$$f(u) = a \qquad \text{if} \quad u < 1\,, \tag{2.1}$$

where $a$ is a given element of $S$, $u \in G$.

Corresponding to each $x \in G$ let there be defined a mapping $\bar{x}$ of $F$ into itself such that

$$(\bar{x}f)(u) = \begin{cases} a & \text{if} \quad u < 1, \\ f(ux) & \text{if} \quad u \geqslant 1\,. \end{cases} \tag{2.2}$$

Lemma 2.1. *For all* $1 \geqslant x \in G$, $(\bar{x}f)(u) = f(ux)$.

Proof. For all $x \leqslant 1$ and $u < 1$, $ux < 1$ so that $(\bar{x}f)(u) = a = f(ux)$.

Theorem 2.2. *If* $x, y \in G$, $\bar{x}\bar{y} = \overline{xy}$ *except when* $x < 1$, $y > 1$ *in which case* $\bar{x}\bar{y} \neq \overline{xy}$.

Proof. If $x \geqslant 1$, $(\bar{x}\bar{y}f)(u) = (\bar{x}f')(u)$ where

$$f'(u) = (\bar{x}\bar{y}f)(u) = \begin{cases} a & \text{if} \quad u < 1, \\ f(uy) & \text{if} \quad u \geqslant 1, \end{cases}$$

so that

$$(\bar{x}\bar{y}f)(u) = (\bar{x}f')(u) = \begin{cases} a & \text{if} \quad u < 1 \\ f'(ux) & \text{if} \quad u \geqslant 1 \end{cases} = (\overline{xy}f)(u)\,,$$

since $u \geqslant 1$, $x \geqslant 1$ implies $ux \geqslant 1$. If $x \leqslant 1$, $y \leqslant 1$, $(\bar{x}\bar{y}f)(u) = (\bar{x}f')(u) = f'(ux)$ by Lemma 2.1 where, $f'(u) = (\bar{y}f)(u) = f(uy)$ again by Lemma 2.1, so that $(\bar{x}\bar{y}f)(u) = f'(ux) = f(uxy) = (\overline{xy}f)(u)$ since $xy \leqslant 1$. Finally if $x < 1$, $y > 1$,

$(\bar{x}\bar{y}f)(u) = (\bar{x}f')(u) = f'(ux)$ by Lemma 2.1 where,

$$f'(u) = (\bar{y}f)(u) = \begin{cases} a & \text{if} \quad u < 1, \\ f(uy) & \text{if} \quad u \geqslant 1, \end{cases}$$

so that

$$(\bar{x}\bar{y}f)(u) = f'(ux) = \begin{cases} a & \text{if} \quad ux < 1, \\ f(uxy) & \text{if} \quad ux \geqslant 1, \end{cases}$$

where as

$$(\overline{xy}f)(u) = \begin{cases} a & \text{if} \quad u < 1, \\ f(uxy) & \text{if} \quad u \geqslant 1. \end{cases}$$

Taking $f \in F$ such that $f(u) \neq a$ for $u \geqslant 1$, it is easily verified that for $u = \max\left((xy)^{-1}, 1\right)$, $(\bar{x}\bar{y}f)(u) = a$ and $(\overline{xy}f)(u) \neq a$, whence it follows that $\bar{x}\bar{y} \neq \overline{xy}$.

Corollary. *If $x \leqslant 1$ then $\bar{x}$ has a left inverse $\bar{x}^{-1}$ and if $y \geqslant 1$ then $\bar{y}$ has a right inverse $\bar{y}^{-1}$.*

Let $K$ be the subsemigroup generated by $\bar{G} = \{\bar{x} \,|\, x \in G\}$ in the semigroup of all mappings of $F$ into itself and let $G_- = \{x \in G \,|\, x \leqslant 1\}$. Then we have

Theorem 2.3. *The elements of $K$ have a unique representation in the form $\bar{x}\bar{y}$, $x \in G_-$, $y \in G_+$.*

Proof. The set $\bar{K} = \{\bar{x}\bar{y} \,|\, x \in G_-, y \in G_+\}$ clearly contains $\bar{G}$ as is seen by taking $x$ or $y = 1$. It is therefore sufficient to prove that $\bar{K}$ is closed under multiplication to show that $\bar{K} = K$. Let $\bar{x}\bar{y}$, $\bar{u}\,\bar{v} \in \bar{K}$. Then since $y \geqslant 1$ we have by a repeated application of Theorem 2.3

$$\bar{x}\bar{y} \cdot \bar{u}\,\bar{v} = \bar{x}\,\overline{yu}\,\bar{v} = \begin{cases} \bar{x}\,\overline{yuv} & \text{if} \quad yu \geqslant 1, \\ \overline{xyu}\,\bar{v} & \text{if} \quad yu \leqslant 1, \end{cases}$$

which proves our assertion. Next we show that if $\bar{x}\,\bar{y} = \overline{uv}$, then $x = u$,

$y = v$. We first show that $x = u$. Assuming the contrary we may without loss of generality suppose that $x < u$. Since $x, u \leqslant 1$, we have

$$(\bar{x}\bar{y}f)(z) = (\bar{y}f)(zx) = \begin{cases} a & \text{if} \quad zx < 1\,, \\ f(zxy) & \text{if} \quad zx \geqslant 1 \end{cases}$$

and similarly

$$(\bar{u}\,\bar{v}f)(z) = \begin{cases} a & \text{if} \quad zu < 1\,, \\ f(zuv) & \text{if} \quad zu \geqslant 1\,. \end{cases}$$

If $f$ is chosen such that $f(v) \neq a$ for $v \geqslant 1$ then for $z = u^{-1}$ we have $zx < 1$ and therefore $(\bar{x}\bar{y}f)(z) = a$, $(\bar{u}\bar{v}f)(z) = f(v) \neq a$, so that $\bar{x}\bar{y} \neq \bar{u}\bar{v}$. Hence we get $x = u$ and so $\bar{x}\bar{y} = \bar{x}\bar{v}$.

Since $x \leqslant 1$, $\bar{x}$ is left invertible with left inverse $\bar{x}^{-1}$ by Corollary to Theorem 2.2. Therefore by multiplying on the left by $\bar{x}^{-1}$ we get $\bar{y} = \bar{v}$. Assuming $y \leqslant v$ so that $yv^{-1} \leqslant 1$ we get again by Corollary to Theorem 2.2 that $\bar{y}\,\bar{v}^{-1} = \bar{v}\,\bar{v}^{-1} = \bar{1}$.

We may write this as $\overline{yv^{-1}}$. $\bar{1} = \bar{1}\cdot\bar{1}$ and applying the result proved above we get $yv^{-1} = 1$ or $y = v$.

Let $\varphi$ be a mapping of $G_+ \times G_+$ into $K$ defined by $\varphi(x, y) = \bar{x}^{-1}\bar{y}$. Then the mapping $\varphi$ is obviously one to one and onto by Theorem 2.3. We may therefore carry over the semigroup structure of $K$ to $G_+ \times G_+$. The multiplication in $G_+ \times G_+$ takes the form

$$(x, y)(u, v) = \bar{x}^{-1}\bar{y}\bar{u}^{-1}\bar{v} = \bar{x}^{-1}\overline{yu^{-1}}\bar{v}$$

$$= \begin{cases} \bar{x}^{-1}\overline{yu^{-1}v} & \text{if} \quad yu^{-1} \geqslant 1 \\ \overline{x^{-1}yu^{-1}}\bar{v} & \text{if} \quad yu^{-1} \leqslant 1 \end{cases} = \begin{cases} (x, yu^{-1}v) & \text{if} \quad y \geqslant u\,, \\ (uy^{-1}x, v) & \text{if} \quad y \leqslant u\,, \end{cases}$$

by the repeated application of Theorem 2.2 for all $(x, y)$, $(u, v)$ in $G_+ \times G_+$.

## References

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## Abstract

*The notion of bibyclic semigroup was first introduced by E. S. Lyapin* [**4**] *and has been studied by Olaf Anderson* [**1**]. *An account of this class of semigroups is also given in Cliffords book* [**2**]. *In our paper* [**3**] *we generalized the concept of a bicycling semigroup formed by* $G_+ \times G_+$ *where* $G_+ = \{x \in G \mid x \geqslant 1\}$, $G$ *being an ordered group, under the multiplication rule*

$$(x, y)(u, v) = \begin{cases} (x, yu^{-1}v), & \text{if } y \geqslant u, \\ (uy^{-1}x, v), & \text{if } y \leqslant u, \end{cases} \quad \text{for all } (x, y),\ (u, v) \text{ in } G_+ \times G_+ .$$

*We showed that for each* $x \in G^+$, $\{(x^m, x^n \mid m, n \geqslant 0\}$ *is a bicyclic subsemigroup of* $G_+ \times G_+$. *Among other properties of the semigroup* $G_+ \times G_+$ *we showed in particular that this special class of semigroup is bisimple inverse. Here in this paper we give a natural representation of the semigroup* $G_+ \times G_+$.

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