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(NB) - property in near-rings (**)

Introduction.

In this paper we study near rings with (NB)-property i.e. near-rings which are nil and of bounded index. It is shown here that $B(N)$, the sum of all the (NB) ideals of N, is a nil ideal, but it need not be (NB). However, it is found that $B(N)$ becomes (NB), if N satisfies the maximum condition of (NB) ideals or minimum condition on nil N-subgroups. Further, $B^*(N)$ is defined as the intersection of all ideals A of N such that N/A contains no non-zero (NB)-ideal. Then $B^*(N)$ is a nil ideal and $N/B^*(N)$ contains no non zero (NB) ideal. Following the construction of Baer lower radical in ring theory, the construction of $B^*(N)$ is given. The effect of d.c.c. on $B(N)$ and $B^*(N)$, and their relation with other radicals is also seen.

1. - $B(N)$ the sum of all (NB) ideals of N.

1.1. – Definition. Let N be a right near-ring. N is said to be (NB): nil and of bounded index, if there exists a *fixed* positive integer $n > 1$ such that $r^n = 0$, for every $r \in N$.

For other definitions, we refer to $[1]_1$, $[1]_2$, [8], [6].

It is clear that nilpotence property implies (NB) property which in turn implies nil property.

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1.2. – Definition. A ideal (N-subgroup) A of N is said to be (NB) *ideal* ($(NB)\,N$-*subgroup*) if there exists a *fixed* positive integer m such that $a^m = 0$ for all a in A.

Remarks. 1. Homomorphic image of an (NB) near-ring (ideal) is again an (NB) near-ring (ideal).

2. Let $\theta: N \to N'$ be a near ring epimorphism such that $\ker\theta$ is an (NB) ideal of N. Then inverse image (under θ) of an (NB) ideal of N' is an (NB)-ideal in N. Hence if $N/\ker\theta$ is an (NB) near-ring, then N is an (NB) near-ring.

1.3. – Theorem. *The sum of two* (NB) *ideals* B_1 *and* B_2 *of* N *is an* (NB) *ideal of* N.

Proof. This is an easy consequence of the near-ring isomorphism $(B_1 + B_2)/B_2 \cong B_1/B_1 \cap B_2$ and the remarks above.

1.4. – Corollary. *The sum of a finite number of* (NB) *ideals is again an* (NB) *ideal.*

Let $B(N)$ denote the sum of all the (NB) ideals of N. In the following we construct an example to show that there exists near-rings for which $B(N)$ need not be (NB) ideal. For this we first take the commutative ring R given in [**3**], (pp. 19-20).

1.5. – Example. Consider the set of symbols X_α where α is any rational number between 0 and 1. Let F be a field of characteristic zero, and R be the commutative algebra over F with $\{X_\alpha \mid 0 < \alpha < 1\}$ as basis. Multiplication of basis element is given by

$$X_\alpha X_\beta = X_{\alpha+\beta} \quad \text{if } \alpha+\beta < 1\,, \qquad X_\alpha X_\beta = 0 \quad \text{if } \alpha+\beta \geq 1\,.$$

R, considered as a ring, is the set of all finite sums $\sum \lambda_\alpha X_\alpha$, where $\lambda_\alpha \in F \cdot R$ is commutative and $R = S(R)$, the sum of all nilpotent ideals of R. Moreover R is nil, but not nilpotent [**3**].

Since every nilpotent ideal is (NB) ideal, we have $R = S(R) \subseteq B(R)$. Hence $R = B(R)$. We claim that $B(R)$ $(= R)$ is not (NB). For if R is (NB), then it must be nilpotent being nil algebra of bounded index over a field of characteristic zero ([**4**], theorem 1.13).

With the help of the above ring R we can always construct a near-ring N which is not a ring and for which $B(N)$ is not (NB). This we can do by taking

$N = R \times N'$, where N' is a near-ring. Then N, together with pointwise addition and pointwise multiplication, is a near-ring. Also $N \cong R \oplus N'$. Since this is a direct sum of ideals, it can be seen that every ideal of R is also an ideal of N. Also, every (NB) ideal of R is (NB) ideal of N. Hence $B(R) \subseteq B(N)$. If $B(N)$ is (NB), then $B(R)$ is also (NB) which is not the case. Hence $B(N)$ is not an (NB) ideal of N.

1.6. – Proposition. *Let E be an ideal of N such that N/E contains no non-zero (NB) ideal, then $B(N) \subseteq E$.*

Proof. $\theta: N \to N/E$ be natural near-ring epimorphism. Then by above remarks, for any (NB) ideal A of N, $\theta(A)$ is an (NB) ideal of N/E. Hence $\theta(A)$ is zero ideal of N/E; that is, $A \subseteq E$. Hence $B(N) \subseteq E$.

We have seen that $B(N)$ may not be (NB) ideal of N. However, we have:

1.7. – Proposition. *$B(N)$ is a nil ideal.*

Proof. Let $U(N)$ denote the sum of all nil ideals of N. Then $N/U(N)$ contains no nonzero nil ideal [**6**], [**7**], and, so, $N/U(N)$ contains no nonzero (NB) ideal. Hence $B(N) \subseteq U(N)$. Since $U(N)$ is nil, $B(N)$ is a nil ideal.

1.8. – Theorem. *If $B(N)$ is (NB) ideal, then $B(N/B(N)) = (0)$.*

Proof. Let $A/B(N)$ be any (NB) ideal of $N/B(N)$. Since $B(N)$ is (NB), A is an (NB) ideal of N. Hence $A \subseteq B(N)$, which means that $N/B(N)$ has no non-zero (NB) ideal. Hence $B(N/B(N)) = (\bar{0})$.

2. - $B^*(N)$ and its construction.

2.1. – Definition. For any near ring N we define

$$B^*(N) = \cap [A \mid A \text{ ideal of } N \text{ such that } N/A \text{ contains no nonzero } (NB) \text{ ideal}].$$

2.2. – Theorem. *$N/B^*(N)$ contains no non-zero (NB) ideal and $B^*(N)$ is a nil ideal.*

Proof. Let $K/B^*(N)$ be an (NB) ideal of $N/B^*(N)$. Then there exists a fixed positive integer n such that $(k + B^*(N))^n = (\bar{0})$ for each $k \in K$. But then $k^n \in B^*(N)$ for each $k \in K$ and for a fixed positive integer n.

If A is any ideal of N such that N/A contains no nonzero (NB) ideal, then $B^*(N) \subseteq A$. Therefore, $k^n \in A$ for each $k \in K$ and a fixed integer $n > 0$. Hence

$(K+A)/A$ is an (NB) ideal of N/A. Since N/A contains no non-zero (NB) ideal, we have that $(K+A)/A = (\bar{0})$; that is, $K \subseteq A$ for every A such that N/A has no nonzero (NB) ideal. Hence $K \subseteq B^*(N)$, which proves that $N/B^*(N)$ has no non-zero (NB) ideal.

To prove that $B^*(N)$ is a nil ideal, we note that $N/U(N)$, where $U(N)$ is the upper nil radical of N, (i.e. sum of all nil ideals of N) contains no non-zero nil ideals and hence contains no non-zero (NB) ideal. Hence $B^*(N) \subseteq U(N)$ and so $B^*(N)$ is a nil ideal.

2.3. – Corollary. $B(N) \subseteq B^*(N)$.

2.4. – Remark. Recall [**6**], [**7**], that the lower nil radical, $L(N)$, of N is the intersection of all ideals A of N such that N/A contains no non-zero nilpotent ideal. Since every nilpotent ideal is (NB), we have that $L(N) \subseteq B^*(N)$. Hence we have the following chains

(a) $S(N) \subseteq B(N) \subseteq B^*(N) \subseteq U(N) \subseteq J_0(N) \subseteq D(N) \subseteq J_1(N) \subseteq J_2(N)$,

(b) $S(N) \subseteq L(N) \subseteq B^*(N) \subseteq U(N) \subseteq J_0(N) \subseteq D(N) \subseteq J_1(N) \subseteq J_2(N)$.

Here $S(N)$ is the sum of all nilpotent ideal of N; $J_\nu(N) = \cap \{[L/N] \mid L\ \nu\text{-modular left ideal of } N\}$ $(\nu = 0, 1, 2)$; and $D(N)$ is the intersection of all modular maximal left ideals of N [**2**], [**6**], [**8**]. If N has identity then it is known that $J_1(N) = J_2(N)$. In this case it is denoted by $J(N)$. Hence for nearings with identity we also have the chains

(c) $S(N) \subseteq L(N) \subseteq B^*(N) \subseteq U(N) \subseteq J_0(N) \subseteq D(N) \subseteq J(N) \subseteq \mathcal{S}^*$,

(d) $S(N) \subseteq L(N) \subseteq B^*(N) \subseteq U(N) \subseteq J_0(N) \subseteq D(N) \subseteq \mathcal{D}^*$,

where $\mathcal{D}^*$ is a radical ideal and $\mathcal{S}^*$ is a strong radical ideal of N [**1**]$_2$, [**6**].

2.5. – *Construction of* $B^*(N)$. We now give the construction of $B^*(N)$ on the lines of the construction of Baer lower radical in ring theory.

Let N be a near-ring. Since $B(N)$ is not necessarity (NB), $N/B(N)$ may contain (NB) ideals. Let N_1 be the ideal of N such that $N_1/B(N)$ is the sum of all (NB) ideals of $N/B(N)$. In general, for every ordinal α, which is not a limit ordinal, we define N_α to be the ideal of N such that $N_\alpha/N_{\alpha-1}$ is the sum of all (NB) ideals of $N/N_{\alpha-1}$. If α is a limit ordinal, we define $N_\alpha = \sum_{\beta<\alpha} N_\beta$, whenever N_β is defined. In this way we obtain an ascending chain of ideals

$$B(N) \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_\alpha \subseteq \ldots .$$

If the set N has ordinal number ν, then after atmost ν steps this chain must stop. We may then consider the (smallest) ordinal τ such that $N_\tau = N_{\tau+1} = \ldots$. We claim that $N_\tau = B^*(N)$.

2.6. – Theorem. $B^*(N) = N_\tau$.

Proof. From the construction it is clear that N/N_τ contains no non-zero (NB) ideal. Therefore $B^*(N) \subseteq N_\tau$.

Conversely, let A be any ideal in N such that N/A contains no non-zero (NB) ideal. Then $B(N) \subseteq A$. By transfinite induction assume that $N_\beta \subseteq A$ for every $\beta < \alpha$. If α is a limit ordinal, then $N_\alpha = \sum_{\beta<\alpha} N_\beta \subseteq A$.

If α si not a limit ordinal, then $\alpha - 1$ exists and $N_{\alpha-1} \subseteq A$. We prove that $N_\alpha \subseteq A$. If $N_\alpha \not\subseteq A$, then there exists an (NB) ideal $C/N_{\alpha-1}$ of $N/N_{\alpha-1}$ such that $C/N_{\alpha-1} \not\subseteq A/N_{\alpha-1}$ (for, if all (NB) ideals of $N/N_{\alpha-1}$ are contained in $A/N_{\alpha-1}$, then $N_\alpha/N_{\alpha-1} \subseteq A/N_{\alpha-1}$, and so, $N_\alpha \subseteq A$, which is not the case). Hence $C \not\subseteq A$. Since $C/N_{\alpha-1}$ is (NB) ideal of $N/N_{\alpha-1}$, we have $\lambda^m \subseteq N_{\alpha-1}$ for every λ in C and a fixed positive integer m. This gives that $\lambda^m \in A$ for every λ in C and a fixed positive integer m. Hence $(C + A)/A$ is a non-zero (NB) ideal of N/A. This contradicts the fact that N/A has no non-zero (NB) ideal. Hence $N_\alpha \subseteq A$ for every α. Hence $N_\tau \subseteq A$ for every A such that N/A has no non-zero (NB) ideal. Hence $N_\tau \subseteq B^*(N)$.

3. - Chain conditions and $B(N)$, $B^*(N)$.

We first show that if N has A.C.C. on (NB) ideals then $B(N) = B^*(N)$. For this we prove:

3.1. – Theorem. *If N satisfies the maximum condition on (NB) ideals, then $B(N)$ is the largest (NB) ideal of N.*

Proof. Since N satisfies the maximum condition on (NB) ideals, there exists a maximal (NB) ideal, say D which is the largest (NB) ideal of N. We show that $B(N) = D$. Clearly, $D \subseteq B(N)$. Now, let $y \in B(N) = \sum_{\alpha \in I} B_\alpha$, where B_α's are (NB) ideals of N. Then $y = b_{\alpha_1} + b_{\alpha_2} + \ldots + b_{\alpha_k}$, $b_{\alpha_i} \in B_{\alpha_i}$ $(1 \leqslant i \leqslant k)$. Also $B_{\alpha_1} + B_{\alpha_2} + \ldots + B_{\alpha_k} + D = D$, since the left hand side is an (NB) ideal containing D and D is a maximal (NB) ideal. Hence $y = b_{\alpha_1} + + \ldots + b_{\alpha_k} \in D$ which gives that $B(N) \subseteq D$. Hence $B(N) = D$.

3.2. – Corollary. *If N has maximum condition on (NB) ideals, then $B(N) = B^*(N)$.*

Proof. By 3.1, $B(N)$ is (NB) ideal and so $B(N/B(N)) = (\bar{0})$. Hence $N/B(N)$ contains no non-zero (NB) ideal. Hence $B^*(N) \subseteq B(N)$. Since $B(N)$ is always contained in $B^*(N)$ we have that $B(N) = B^*(N)$.

3.3. – Theorem. *If N satisfies* d.c.c. *on (NB) N-subgroups, then every (NB) N-subgroup is nilpotent.*

Proof. Let C be any (NB) N-subgroup of N. For each positive integer n, let C^n denote the N-subgroup of N generated by the set $C^n = \{c_1 c_2 \ldots c_n | |c_i \in C\}$. Then $C = C^1 \supseteq C^2 \supseteq C^3 \supseteq \ldots C^n \supseteq C^{n+1} \supseteq \ldots$ is decreasing sequence of (NB) N-subgroups of N. Since N has d.c.c. on (NB) N-subgroups, there exists a positive integer k such that $C^k = C^{k+1} = \ldots = C^{2k} = \ldots$. Put $B = C^k$. We show that $B = (0)$. Let us assume that $B \neq (0)$. Also let $B \circ B$ be the N-subgroup of N generated by the set $BB = \{b_1 b_2 | b_1, b_2 \in B\}$. Since B is an N-subgroup of N, it contains BB, and therefore: $B \supseteq B \circ B = C^k \circ C^k \supseteq C^{2k} = C^k = B$. This gives that $B \circ B = B$ and so $BB \neq 0$. Now consider the collection

$$\mathscr{F} = \{L | L(NB) \ N\text{-subgroup of } N \text{ such that } L \subseteq B \text{ and } BL \neq 0\} .$$

Since $B \in \mathscr{F}$, we have that $\mathscr{F}$ is not empty. Since N satisfies d.c.c. on (NB) N-subgroups, $\mathscr{F}$ contains a minimal element, say D. So $D \subseteq B$ and $BD \neq 0$. Thus, there exists a $d \in D$ such that $Bd \neq 0$. Also Bd is an (NB) N-subgroups of N contained in B, for $Bd \subseteq BD \subseteq BB \subseteq NB \subseteq B$ and B is (NB). Moreover $N(Bd) \neq 0$. To see this, consider the map $\theta\colon N \to N$ given by $\theta(r) = rd$ for all r in N. Clearly, θ is an N-homomorphism. If $B(Bd) = 0$, then $(BB)d = 0$ and so $BB \subseteq \ker\theta$. This implies that $B \circ B \subseteq \ker\theta$ and so $B \subseteq \ker\theta$, which means $\theta(B) = Bd = 0$. Hence $B(Bd) \neq 0$. So $Bd \in \mathscr{F}$. Since D is a minimal element of $\mathscr{F}$ we have that $Bd = D$. So $d = bd$ for some $b \in B$. Since B is (NB), there exists a (fixed) m such that $b^m = 0$ for all $b \in B$. Hence $d = bd = b^2 d = b^3 d = \ldots = b^m d = 0$ which is a contradiction. Hence $B = C^k = 0$ and C is nilpotent.

3.4. – Corollary. *If N satisfies* d.c.c. *on (NB) N-subgroups then $B(N) = S(N)$, the sum of all nilpotent ideals of N.*

Proof. We always have $S(N) \subseteq B(N)$. Since N has d.c.c. on (NB) N-subgroups, every (NB) ideal is nilpotent. Hence $B(N) \subseteq S(N)$.

Laxton ([**5**], theorem 2.6) proved that if N is a d.g. near-ring with identity which satisfies the d.c.c. on N-subgroups, then every nil N-subgroup is nilpotent. This was slightly generalised by Beidleman, who proved [1]$_2$ that if N is a d.g. near-ring with identity which satisfies the d.c.c. on nil N-subgroups, then every nil N-subgroup is nilpotent. The following theorem shows that the hypothesis of d.g. ness with identity in the above cited result is not necessary.

3.5. – Theorem. *Let N be a near-ring satisfying* d.c.c. *on nil N-subgroups, then every nil N-subgroup is nilpotent.*

Proof. Replace (NB) by nil in the proof of the Theorem 3.3.

Since $U(N)$ is a nil ideal, the above theorem together with the Remark 2.4 gives.

3.6. – Corollary. *Let N satisfy* d.c.c. *on nil N-subgroups, then $B(N)$ and $B^*(N)$ are nilpotent ideals, and $S(N) = B(N) = B^*(N) = L(N) = U(N)$.*

Since every nil N-subgroup is left quasi regular, we have that d.c.c. on left quasi regular N-subgroups implies d.c.c. on nil N-subgroups.

Hence by ([6], 6.8(c)) and above corollary we have that:

3.7. – Corollary. *Let N satisfy* d.c.c. *on left quasi regular N-subgroups, then $S(N) = B(N) = L(N) = B^*(N) = U(N) = J_0(N)$.*

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Summary

See Introduction.

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